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Existence of Exact Zero Divisors and Totally Reflexive Modules in Artinian Rings

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EXISTENCE OF EXACT ZERO DIVISORS AND TOTALLY REFLEXIVE MODULES IN ARTINIAN RINGS

by

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Abstract

EXISTENCE OF EXACT ZERO DIVISORS AND TOTALLY REFLEXIVE MODULES IN ARTINIAN RINGS

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In this dissertation, we consider commutative local Artinian rings $(A, \mathfrak{m}, k)$, which are rings that satisfy the descending chain condition on ideals. First, we investigate the existence of exact zero divisors in Artinian Gorenstein rings. We say that a pair of elements a, b in $\mathfrak{m}$ is an exact pair of zero divisors in A if $\operatorname{ann}_A(a) = (b)$ and $\operatorname{ann}_A(b) = (a)$. It is known that a generic Artinian Gorenstein ring of socle degree 3 contains at least one pair of exact zero divisors. We are interested in the existence of exact pairs of zero divisors in the case of socle degree bigger than 3. We present the conditions when an Artinian Gorenstein ring of socle degree bigger than 3 contains linear pairs of exact zero divisors.

We also investigate the existence of totally reflexive modules in the absence of exact pairs of zero divisors. Since the existence of totally reflexive modules is guaranteed in Artinian Gorenstein rings, we consider Artinian Non-Gorenstein rings. As a result, we obtain a class of rings having non-free totally reflexive modules in the absence of exact pairs of zero divisors. We also discuss the Weak Lefschetz property and

the connection between the Weak Lefschetz Property and exact pairs of zero divisors. We use Macaulay's Inverse System to construct rings in both investigations.

Table of Contents

Chapter 1

INTRODUCTION

Besides this introduction, this dissertation consists of four additional chapters containing some preliminary definitions and the proofs of main results. This reseach lies within the subject area of commutative and homological algebra. The particular focus of this thesis is Artinian rings, those rings that satisfy the descending chain condition on ideals. They are also called Artin rings and are named after Emil Artin, who first discovered that the descending chain condition for ideals simultaneously generalizes finite rings and rings that are finite-dimensional vector spaces over fields.

This thesis work has two main components. The first part investigates the existence of exact pairs of zero divisors. The second part investigates the existence of totally reflexive modules in the absence of exact pair of zero divisors. These investigations take place specifically over Artinian Gorenstein and Artinian non-Gorenstein rings, respectively. These rings are also finite dimensional vector spaces over their coefficient field. The motivation to study exact pairs of zero divisors and totally reflexive modules comes from the fact that one can construct totally reflexive modules from exact pairs of zero divisors. In fact, if a and b form an exact pair of zero divisors in a ring A then the modules $(a) \cong A/(b)$ and $(b) \cong A/(a)$ are totally reflexive. These modules are of interest since they are essential objects in Gorenstein relative homological algebra and are used to extend the relative Ext and relative Tor derived functors into negative homological degrees. We also discuss the connection between exact pairs of zero divisors and the Weak Lefschetz property.

In Chapter 2, we discuss the definitions and notation that will be used throughout this thesis. For more detail on Artinian rings and Gorenstein rings, see [8,31], and for the areas of homological algebra, we will be following the definitions and notation of [32,36].

In Chapter 3, we present Macaulay's Inverse System, which we use as a tool to construct both Artinian Gorenstein and Artinian non-Gorenstein rings. The study of the inverse system requires us to introduce injective modules, injective envelope, and Matlis duality. Macaulay's Inverse System [30] tells us that there is a one-to-one correspondence between finitely generated nonzero R-submodules F in $S = K[y_1, y_2, \ldots, y_e]$ and ideals $I = \text{ann}_R(F)$ in R, where $\text{ann}_R(F) = \{r \in R : ry = 0 \forall\, y \in M\}$, and R/I is a local Artinian ring. If we take F to be a cyclic module, then the corresponding Artinian ring becomes Gorenstein.

In Chapter 4, we consider only Artinian Gorenstein rings and establish results about the existence of exact pair of zero divisors in these rings. Let $R = k[x_1, x_2, \ldots, x_e]$ be a polynomial ring with coefficients in a field k, and let $I \subset R$ be a homogeneous ideal. The graded ring $A = R/I$ is then Artinian if there exists a positive integer $d > 0$ such that $A_{d+1} = 0$. An Artinian ring with the maximal ideal $\mathfrak{m}$ is called Gorenstein if $\text{ann}_A(\mathfrak{m})$ is one-dimensional as a vector space over k. If $d+1$ is the least integer such that $A_{d+1} = 0$ and if A is Gorenstein, then $Soc(A) = A_d$. In this instance, d is called the socle degree of A. It seems that the notion of exact pairs of zero divisors first appeared in [27] under the name of exact sequences of pairs. Later they were reintroduced by Henriques and Sega as exact pairs of zero divisors in [21], and recently they have been studied widely, see [6], [12], [22] and [28]. A pair of non zero elements $a, b \in A$ is said to be an exact pair of zero divisors if $\text{ann}_A(a) = (b)$ and $\text{ann}_A(b) = (a)$. Their existence is known for rings with socle degree 3, and our interest in higher socle degrees has led to some new results and techniques. Our different

approach allows us to extend results to the higher socle degree case. The proof of the main theorem heavily relies on so-called catalecticant matrices and the varieties defined by the maximal minors of these matrices and annihilators of associated matrices. The approach that we use is to show that the condition of these rings having an exact pair of zero divisors corresponds to a non-empty Zariski open subset in some projective variety. We prove that for Artinian Gorenstein rings of socle degree $d > 3$, having an exact pair of zero divisors is an open condition under a nontrivial closed condition.

In the late 1960s, Auslander and Bridger first introduced the idea of totally reflexive modules in [3], but the terminology "totally reflexive" was introduced in 2002 by Avramov and Martsinkovsky [4]. Totally Reflexive modules have been studied by many researchers using several different terminologies, such as modules of G-dimension zero [3], maximal Cohen-Macaulay modules [9], and (finitely generated) Gorenstein-projective modules [18]. The existence of totally reflexive modules is always guaranteed, since every free module is trivially totally reflexive. It is also known that over a Gorenstein local ring, the totally reflexive A-modules are exactly the maximal Cohen-Macaulay modules. Thus we are interested in observing their existence over non-Gorenstein local ring. There has been some work done to answer this existence question over non-Gorenstein rings, see [12], [2] and [37]. Most of the recent constructions of totally reflexive modules start with an exact pair of zero divisors (a, b), see for example [12]. If (a, b) is a pair of exact zero divisors, then the complex

$$\cdots \to A \xrightarrow{a} A \xrightarrow{b} A \xrightarrow{a} \cdots$$

is a totally acyclic complex and $A/(a)$ and $A/(b)$ are totally reflexive modules. However, it is mentioned in [12] that non-free totally reflexive modules may exist even in the absence of an exact pair of zero divisors. Christensen, Jorgensen, Rahmati,

Striuli, and Wiegand proved in [12] that a generic standard graded k-algebra A with Hilbert series $1+et+(e-1)t^2$ has an exact pair of zero divisors and a non-free cyclic totally reflexive module. They also found an example of a ring over field of characteristic 2 that admits non-free totally reflexive modules, but does not have an exact pair of zero divisors. In a recent study, Vraciu and Atkins constructed an example of ring having codimension 8 that has totally reflexive modules in the absence of exact pairs of zero divisors. These results motivated us to look for classes of Artinian rings, possibly of smaller codimension, that admit totally reflexive modules in the absence of exact pairs of zero divisors. We were able to obtain totally reflexive modules in the absence of exact pairs of zero divisors in rings of codimension 5 and higher, Theorem 5.2.5.

In Chapter 6, we provide the definition of the Weak Lefschetz property, and we present some well-known results about this property. We also establish a connection between the Weak Lefschetz property and exact pairs of zero divisors in certain Artinian rings. The Weak Lefschetz property simply means that there exists a linear form $a \in A$ such that the multiplication map $A_i \xrightarrow{a} A_{i+1}$ has maximal rank for every i. A linear form having maximal rank is equivalent to it having a minimal annihilator, and linear forms with minimal annihilators are candidates for being exact zero divisors. We conjecture that an Artinian quadratic algebra $A = k[x_1, \ldots, x_e]/I$ with homogeneous ideal I and Hilbert series $1 + et + (e-1)t^2$ has the Weak Lefschetz property if and only if A admits an exact pair of zero divisors.

Chapter 2

DEFINITIONS AND PRELIMINARY CONCEPTS

In this chapter, we present the basic definitions that will be used throughout this thesis. All rings will be commutative. The notation $(A, \mathfrak{m}, k)$ means that A is a local ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. All modules are assumed to be finitely generated.

2.1 Artinian Rings

Definition 2.1.1. A ring A is a *Noetherian ring* if it satisfies the ascending chain condition for ideals. That is, for any increasing sequence of ideals $I_1 \subseteq I_2 \subseteq I_3 \cdots$, there exists a natural number n such that $I_n = I_{n+1} = I_{n+2} = \cdots$. A module M is a *Noetherian module* if it satisfies ascending chain condition on it's submodules. It is equivalent to say that every submodule of M is finitely generated. The name "Noetherian" comes from the mathematician Emmy Noether.

Definition 2.1.2. A ring A is an *Artinian ring* if it satisfies the descending chain condition for ideals. That is, for any decreasing sequence of ideals $I_1 \supseteq I_2 \supseteq I_3 \cdots$, there exists a natural number n such that $I_n = I_{n+1} = I_{n+2} = \cdots$. They are also called Artin rings and are named after Emil Artin. A module M is an *Artinian module* if it satisfies descending chain condition on it's submodules.

Example 2.1.3.

1. The rings $\mathbb{Q}$ and $\mathbb{Z}/n\mathbb{Z}$ are both Noetherian and Artinian rings.

2. The rings $\mathbb{Z}$ and $k[x]$ are Noetherian but not Artinian. Let I be an ideal in $\mathbb{Z}$. Since every ideal in $\mathbb{Z}$ is principal, $I = n\mathbb{Z}$ where n is some integer. Observe that for any $n > 1$, we have $(n) \supset (n^2) \supset (n^3) \supset \dots$ which is a decreasing chain of ideals that does not stabilize. Hence $\mathbb{Z}$ is not an Artinian ring. The descending chain of ideals $(x) \supset (x^2) \supset (x^3) \supset \dots$ in $k[x]$ never stabilizes either.
3. The polynomial ring $k[x_1, x_2, x_3, \dots]$ in infinitely many variables is neither Artinian nor Noetherian. The sequence of ideals $(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \dots$ is ascending, and does not stabilize. Thea chain $(x_1) \subset (x_1^2) \subset (x_1^3) \subset \dots$ is ascending and never stabilizes.
4. Suppose A is a finite-dimensional k-algebra, and M is a finitely generated A-module. Then M is Artinian.
5. A vector space V over a field k is Artinian as a k-module if and only if it is a finite-dimensional over k.

Definition 2.1.4. The *socle* of a local ring $(A, \mathfrak{m}, k)$ is the ideal $\mathrm{Soc(A)} = \mathrm{ann}_A(\mathfrak{m}) = \{r \in A \mid r\mathfrak{m} = 0\}$.

Example 2.1.5.

1. If $A = k[x, y]/(x^2, xy, y^2)$, then $\mathrm{Soc}(A)$ is the maximal ideal $\mathfrak{m} = (x, y)$.
2. If $A = k[x, y, z]/(xy, yz, xz, x^2 - y^2, y^2 - z^2)$, then $\mathrm{Soc}(A) = (x^2)$.
3. If $A = k[x, y]/(x^2, xy, y^3)$, then $\mathrm{Soc}(A) = (x, y^2)$.

2.2 Hom Functor

In this section we recall the definition of the Hom functor.

Definition 2.2.1. Let M and N be two A-modules. The set $\mathrm{Hom}_A(M,N)$ of all A-module homomorphisms $f : M \to N$ is an A-module under natural addition and scalar multiplication.

In the special case where $N = A$, we get the A-module $M^* = \mathrm{Hom}_A(M,A)$. This is called the *dual module, or A-dual of M*. In particular, $A^* = \mathrm{Hom}_A(A,A)$ is isomorphic to A.

Suppose $g : M' \to M$ is an A-module homomorphism. Define

$$\mathrm{Hom}_A(g,N) : \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M',N)$$

by $\mathrm{Hom}_A(g,N)(f) = f \circ g$ for $f \in \mathrm{Hom}_A(M,N)$. Note that $\mathrm{Hom}_A(-,N)$ reverses the direction of the arrow, as seen by the following computation; for $h : N \to M'$,

$$\mathrm{Hom}_A(g \circ h, N)(f) = f \circ (g \circ h) = (f \circ g) \circ h = \mathrm{Hom}_A(h,N)(f \circ g)$$

$$= \mathrm{Hom}_A(h,N)(\mathrm{Hom}_A(g,N)(f))$$

$$= \big(\mathrm{Hom}_A(h,N) \circ \mathrm{Hom}_A(g,N)\big)(f).$$

We say that $\mathrm{Hom}_A(-,N)$ is a *contravariant functor* on the category of A-modules.

Proposition 2.2.2. *For a local ring* $(A,\mathfrak{m},k)$, $\mathrm{Soc}(A) \cong \mathrm{Hom}_A(k,A)$.

Proof. Recall that $\mathrm{Soc}(A) = \mathrm{ann}(\mathfrak{m})$ and $A/\mathfrak{m} = k$. Define a map

$$\phi : \mathrm{Hom}_A(k,A) \to \mathrm{ann}_A(\mathfrak{m})$$

by $\phi(g) = g(\bar{1})$. If $x \in \mathfrak{m}$, then $x.g(\bar{1}) = g(\bar{x}) = g(\bar{0}) = 0$. Therefore $g(\bar{1}) \in \mathrm{ann}_A(\mathfrak{m})$, so the map makes sense. It is easy to see that ϕ is a homomorphism. Let $g \in \ker(\phi)$. Then $g(\bar{1}) = 0$ and so $g = 0$. This implies that $\ker(\phi) = 0$.

To show the surjectivity of ϕ, it is enough to show that there exists $g \in \mathrm{Hom}_A(k,A)$ such that $\phi(g) = x$ for every $x \in \mathrm{ann}_A(\mathfrak{m})$. If we define $g : k \to A$ by $g(\bar{1}) = x$, then g is a well-defined homomorphism and this completes our proof. $\square$

Proposition 2.2.3. *(Nakayama's Lemma) Let A be a local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated A-module. If $\mathfrak{m}M = M$, then $M = 0$.*

Fact 2.2.4.

1. $\mathrm{Soc}(A)$ is a vector space over $R/\mathfrak{m}$.
2. If A is a local Artinian ring, then its socle is nonzero.

Proof. If $\mathfrak{m}$ is the maximal ideal of A, then we have the descending chain $\mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots$. Since A is Artinian, there exists a smallest number $i \geq 0$ such that $\mathfrak{m}^i = \mathfrak{m}^{i+1}$. Therefore $\mathfrak{m}.\mathfrak{m}^i = \mathfrak{m}^i$, and Nakayama's Lemma implies that $\mathfrak{m}^i = 0$. Since $\mathfrak{m}^{i-1} \neq \mathfrak{m}^i$, we have $\mathfrak{m}^{i-1} \neq 0$, and therefore $\mathfrak{m}^{i-1} \subseteq \mathrm{Soc}(A)$. □

2.3 Injective Modules and Matlis Duality

In this section, we present the basic concepts that are needed to introduce Matlis duality.

Definition 2.3.1. An A-module E is *injective* if and only if $\mathrm{Hom}_A(-, E)$ is exact. Recall that $\mathrm{Hom}_A(-, E)$ is always left exact, Thus an A-module is injective if and only if $\mathrm{Hom}_A(-, E)$ is right exact.

Definition 2.3.2. The *injective resolution* of an A-module M is an exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots,$$

where the E_n are inective modules over A.

Definition 2.3.3. The *injective dimension* of an A-module M is the smallest integer $n \geq 0$ such that there is an injective resolution of the form

$$0 \to M \to E_0 \to \cdots \to E_n \to 0.$$

Our notation for injective dimension is $\mathrm{injdim}_A(M)$.

Definition 2.3.4. An A-module N is said to be an *essential extension* of M if $M \subset N$, and for every submodule N_0 of N, $N_0 \cap M \neq 0$. Such an essential extension is called *maximal* if no module properly containing N is an essential extension of M. A maximal essential extension of a module always exists and is unique up to isomorphism.

Definition 2.3.5. It is known that any module M can be embedded into an injective module. If the embedding chosen is minimal, the corresponding injective module is called the *injective hull* of M and is unique up to isomorphism, but the isomorphism is not necessarily unique [8, 3.2]. In this case E is an essential extension of M. Our notation for the injective hull of M will be $E(M)$ or $E_A(M)$.

Let (A, m, k) be a complete Noetherian local ring, and let E be the injective hull of k. The the functor $\mathrm{Hom}_A(-, E)$ establishes an anti-equivalence between the category of Artinian A-modules and the category of Noetherian A-modules. This result is known as *Matlis duality*.

Definition 2.3.6. The *Matlis dual* of an A-module M is the module

$$M^\vee := \mathrm{Hom}_A(M, E_A(k)).$$

With this definition, we can write $(-)^\vee = \mathrm{Hom}_A(-, E_A(k))$, which is a contravariant exact functor from the category of A-modules to itself.

If there is a finite sequence $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ of submodules of M such that each quotient module M_{i+1}/M_i is simple and nonzero for all $i = 0, 1 \ldots n-1$, then we define n to be the *length* of M. If such a finite sequence does not exist, then the length of M is defined to be ∞. Define $l(M) :=$ length of M.

Corollary 2.3.7. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. Then, for every finite length module M, $l(M) = l(M^\vee)$.*

Proof. We use induction on $l(M)$. Observe that $k^\vee = \mathrm{Hom}_A(k, E) \cong \mathrm{Soc}(E)$, since $\mathrm{Soc}(E)$ is one-dimensional, we have $k^\vee \cong k$. If $l(M) = 1$, then $M \cong k$ and $l(M^\vee) = l(k^\vee) = l(k) = 1$. Now suppose $l(M) > 1$. Then there exists an exact sequence

$$0 \to k \to M \to N \to 0$$

where $l(N) = l(M) - 1$. Since E is injective, the sequence

$$0 \to N^\vee \to M^\vee \to k^\vee \to 0$$

is exact. By induction hypothesis, $l(M^\vee) = l(N^\vee) + 1 = l(N) + 1 = l(M)$. □

Theorem 2.3.8. *Let $(A, \mathfrak{m}, k)$ be an Artinian local ring and let $(-)^\vee$ denote $\mathrm{Hom}_A(-, E)$, where $E = E_A(k)$. Then $l(E_A(k)) = l(A)$.*

Proof. Since A is an Artinian, it has finite length as an A-module. By using the previous corollary, we have $A^\vee = \mathrm{Hom}_A(A, E_A(k)) \cong E_A(k)$ and hence $l(A) = l(A^\vee) = l(E_A(k))$. □

Matlis duality states that the duality functor $^\vee$ gives an anti-equivalence between {Artinian A-modules} $\longleftrightarrow$ {Noetherian A-modules}. In particular the duality functor gives an anti-equivalence from the category of finite-length modules to itself.

Theorem 2.3.9. [8, Theorem 3.2.13] *Let $(A, \mathfrak{m}, k)$ be a complete Noetherian local ring, let $E = E_A(k)$, and let M be an A-module. Then:*

(i) If M is Noetherian, then $M^\vee$ is Artinian.

(ii) If M is Artinian, then $M^\vee$ is Noetherian.

(iii) If M is either Artinian or Noetherian, then $M^{\vee\vee} \cong M$.

2.4 Artinian Gorenstein Rings

There are many equivalent definitions of a Gorenstein ring, and some of them are listed below. One commonly used definition is the following:
A Noetherian local ring A is *Gorenstein* if it has finite injective dimension as an A-module. It is well known that, for an n-dimensional Noetherian ring A, $\text{injdim}(A) < \infty \Leftrightarrow \text{injdim}(A) = n$.

Proposition 2.4.1. *Let $(A, \mathfrak{m}, k)$ be an Artinian local ring. Then the following statements are equivalent.*

(i) A is Gorenstein

(ii) A is self-injective

(iii) The socle of A is one-dimensional.

Proof. $(i) \Leftrightarrow (ii)$. Let A be Gorenstein. Since A is Artinian, $\dim(A) = 0 \Leftrightarrow \text{injdim}(A) = 0 \Leftrightarrow A$ is self injective.

$(ii) \Rightarrow (iii)$. Since A is Artinian, there exists an exact sequence $0 \to k \to A$. The sequence $\text{Hom}_A(A, A) = A \to \text{Hom}_A(k, A) \to 0$ is exact because A is injective. Therefore, $\text{Hom}_A(k, A)$ is generated by one element, and $\text{Hom}_A(k, A) \neq 0$. Thus we must have $\text{Hom}_A(k, A) \cong \text{Soc}(A)$ is one-dimensional.

$(iii) \Rightarrow (ii)$. Assume that $\text{Soc}(A)$ is one-dimensional, i.e. $\text{Soc}(A) \cong k$, and let $E = E_A(k)$ be the injective hull of the residue field of A. Every 0 dimensional ring is an essential extension of it's socle. Given any submodule I, that is an ideal of A, there is a smallest integer i such that $\mathfrak{m}^i I = 0$ and $\mathfrak{m}^{i-1} I \subseteq \text{Soc}(A) \cap I \neq 0$. But by Theorem 2.3.8, E and A have the same length, thus we must have $E \cong A$. Hence A is injective as required. □

As we can see, Gorenstein rings are defined in different ways. Since our interest is in the Artinian case, we will be using the following definition throughout this thesis.

Definition 2.4.2. An Artinian local ring $(A, \mathfrak{m}, k)$ is called *Gorenstein* if its socle is a one-dimensional k-vector space.
The *socle degree* of an Artinian local ring A is the greatest integer d such that $\mathfrak{m}^d \neq 0$. There exists such d by Fact 2.2.4 (ii).

Example 2.4.3.

1. Let $A = k[x, y, z]/(x^2, y^2, xz, yz, z^2 - xy)$. Then A is a Gorenstein ring with socle degree 2, where $\mathrm{Soc}(A) = kz^2$.
2. Let $A = k[x, y]/(x^2, y^2, xy)$. Then A is not Gorenstein because $\mathrm{Soc}(A) = (x, y)$, which is not a one-dimensional vector space.

Definition 2.4.4. The *depth* of an module R-module M is the maximal integer s such that there is a regular sequence in M of length s. The module M is called a *Cohen-Macaulay module* if $\mathrm{depth}(M) = \dim(M)$. If $\mathrm{depth}(M) = \dim(M) = \dim(A)$, then the module is called a *maximal Cohen-Macaulay module, or MCM*. If A itself is a Cohen Macaulay module then it is called *Cohen Macaulay ring*.

2.5 Exact Pairs of Zero Divisors

Definition 2.5.1. Let A be a ring. A pair of non zero elements (a, b) in A is said to be an *exact pair of zero divisors* if $\mathrm{ann}_A(a) = (b)$ and $\mathrm{ann}_A(b) = (a)$. An element $a \in A$ is an exact zero divisor if it belongs to a pair (a, b) of exact zero divisors. Equivalently, it satisfies the condition

$$R \neq \mathrm{ann}(b) \cong R/(a)$$

and

$$R \neq \operatorname{ann}(a) \cong R/(b).$$

If a and b are linear forms in A, then we call (a, b) *linear exact pair of zero divisors.*

Example 2.5.2.

1. Let $A = k[x, y]/(x^2 - y^2, xy)$. The pair (x, y) is an exact pair of zero divisors. The pair $(x - y, x + y)$ is also an exact pair of zero-divisors.
2. Let $A = k[x, y]/(x^2, xy)$. Here, (x, y) is not an exact pair of zero-divisors, since $\operatorname{ann}_A(y) = (x)$, but $\operatorname{ann}_A(x) = (x, y) \neq (y)$.

Definition 2.5.3. The *Hilbert function* of a graded ring A is the numerical function defined by

$$H_A(t) = \dim_K(A_t)$$

with $t \in \mathbb{N}$.

The *Hilbert series* of A is the formal power series $H_A(t) = \sum_n (\dim_k(A_n))t^n$. If A is also Artinian, then the Hilbert series of A is actually a polynomial.

2.6 Complexes and Totally Reflexive Modules

Definition 2.6.1. A *complex* is a sequence of A-modules and homomorphisms,

$$C: \qquad \cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots,$$

such that $d_i d_{i+1} = 0$ for all $i \in \mathbb{Z}$. Equivalently, $\operatorname{Im}(d_{i+1}) \subset \ker(d_i)$ for all i. A complex is *exact (acyclic)* if $\ker(d_i) = \operatorname{Im}(d_{i+1})$ for all i.

Definition 2.6.2. Let M be an A-module. A *free resolution* of M is an exact complex

$$\mathbb{F}: \qquad \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

in which each F_i is a free A-module. If $\mathbb{F}$ is a free resolution of M, then its *deleted free resolution* is the complex of the form

$$\mathbb{F}: \qquad \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0.$$

No information is lost by omitting M since $M \cong \mathrm{Coker}(F_1 \xrightarrow{d_1} F_0)$.

Definition 2.6.3. A complex $\mathbb{F} = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ of finitely generated free A-modules is *totally acyclic* if both the complex $\mathbb{F}$ and the dual complex $\mathbb{F}^* = \mathrm{Hom}_A(\mathbb{F}, A)$ are exact. An A-module M is said to be *totally reflexive* if $M \cong \mathrm{Coker}(F_1 \xrightarrow{d_1} F_0)$ for some totally acyclic complex $\mathbb{F}$.

The notion of totally reflexive modules is due to Auslander and Bridger [3]. They were introduced as modules of Gorenstein dimension zero. These modules were used as a generalization of free modules, in order to define a new homological dimension for finitely generated modules over Noetherian rings, called the G-dimension. Over a Gorenstein ring, the totally reflexive modules are exactly the maximal Cohen-Macaulay modules, and Gorenstein rings are characterized by the fact that every finitely generated module has finite G-dimension. All totally reflexive modules have a doubly infinite resolution of free modules, which we call a complete resolution.

Definition 2.6.4. The *Gorenstein dimension* of a module M denoted, $\text{G-dim}_A(M)$, is the smallest integer $n \geq 0$, such that there exists an exact sequence

$$0 \to T_n \to T_{n-1} \to \ldots \to T_0 \to M \to 0$$

in which each T_i is totally reflexive module for for $0 \geq i \geq 1$. If no such integer exists, then $\text{G-dim}_A(M) = \infty$.

Note that every finitely generated free module M is totally reflexive. Since in this case the complex $0 \to M \to M \to 0$, with M in homological degrees 0 and 1, is totally acyclic. Let A be a ring. If

$$M' \to M \to M'' \to 0$$

is an exact sequence of A-modules, then the induced sequence

$$\mathrm{Hom}_A(-, N): \qquad 0 \to \mathrm{Hom}_A(M'', N) \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N)$$

is exact. Similary, if

$$0 \to N' \to N \to N''$$

is an exact sequence of A-modules, then the induced sequence

$$\mathrm{Hom}(-, N): \qquad 0 \to \mathrm{Hom}_A(M, N') \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N'')$$

is exact. We call the properties of $\mathrm{Hom}(-,-)$ described in the above proposition *left exactness*.

Definition 2.6.5. The *embedding dimension* of an A-module M is $\dim(\mathfrak{m}/\mathfrak{m}^2)$, which is the minimal number of generators of the maximal ideal of A. Our notation for the embedding dimension of A will be $\mathrm{edim}(A)$. The *codimension* of A is the number $\mathrm{codim}(A) = \mathrm{edim}\, A - \dim(A)$.

Definition 2.6.6. A non-zero element in $a \in (A, \mathfrak{m}, k)$ is called a Conca generator if $a^2 = 0$ and $a\mathfrak{m} = \mathfrak{m}^2$.

Chapter 3

MACAULAY'S INVERSE SYSTEM

In this chapter, we introduce Macaulay's Inverse System, which is a special case of Matlis duality. This system is the main tool we use to construct our rings of interest. The concepts in this chapter provide a method which can be used to determine the existence of exact pairs of zero divisors.

3.1 Macaulay's Inverse System

In this section we will consider two polynomial rings R and S at the same time. We assume that k is algebraically closed and that the characteristic of k is 0. Let $R = k[x_1, x_2, \ldots, x_e]$ and $S = k[y_1, y_2, \ldots, y_e]$ be polynomial rings with coefficients in k, and with $deg(x_i) = 1$ and $deg(y_j) = 1$ for all $1 \leq i, j \leq e$. In 1916, Macaulay stated a one-to-one correspondence between Artinian Gorenstein rings $A = R/I$ and polynomials in S, see [30]. This correspondence can be extended to all Artinian rings $A = R/I$, and finitely generated R-submodules of S. It turns out that Macaulays correspondence is a particular case of Matlis duality, see [17, Theorem 2.3 and Proposition 2.4].

Let us now introduce the contraction action, which will play a key role in Macaulay's Inverse System.

Definition 3.1.1. (Contraction Action) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_e)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_e)$ be multi-indices . We denote by x^α and y^β the monomials $x_1^{\alpha_1} \ldots x_e^{\alpha_e}$ and $y_1^{\beta_1} \ldots y_e^{\beta_e}$, respectively.

Define the *contraction action*

$$\circ : R \times S \to S$$

by

$$x^\alpha \circ y^\beta = \left\{ \begin{array}{ll} y^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{otherwise} \end{array} \right\},$$

where $\beta \geq \alpha$ means $\beta_i \geq \alpha_i$ for all $i = 1, 2, \ldots, e$ and $\beta - \alpha$ is the multi-index $(\beta_1 - \alpha_1, \ldots, \beta_e - \alpha_e)$.

We extend this action by linearity to all polynomials of R on S. In this way, the $\{x_i\}$ behave like the basis of R_1 dual to the basis $\{y_i\}$ of S_1. Hence, R_1 can be thought of as the dual space $\mathrm{Hom}_k(S_1, k)$ of S_1, see [26].

Example 3.1.2.

Let $x_1, x_2 \in R$ and $y_1, y_2 \in S$.

1. $x_1 \circ y_1^2 = y_1$
2. $x_2 \circ y_1^2 = 0$
3. $(x_1 x_2) \circ (y_1 y_2) = 1$
4. $x_1 \circ (y_1 y_2) = y_2$

Notice that the action of R on S turns S into an R-module, since, for $r, r_2, r_3 \in R$ and $s, s_1, s_2 \in S$, we have

1. $r \circ (s_1 + s_2) = r \circ s_1 + r \circ s_2$,
2. $(r_1 r_2) \circ s = r_1 \circ (r_2 \circ s)$,
3. $(r_1 + r_2) \circ s = r_1 \circ s + r_2 \circ s$,
4. and $1 \circ s = s$, and
5. If $c \in k$, then $r \circ (cs) = (cr) \circ s = c(r \circ s)$.

When the characteristic of k is zero, this action is closely related to partial differentiation as we described in the next proposition. We can think of the polynomials of R as representing partial differential operators and the polynomials of S as the real polynomials on which the differential operators act. This action is also called the *apolarity action* of R on S, see [26].

Proposition 3.1.3. *Let S' denote the R-module S, except using the partial differentiation action. Then the R-module homomorphism $\phi : S \to S'$ defined by*

$$\phi(y^\alpha) = \frac{1}{\alpha!} y^\alpha,$$

where $\alpha! = \alpha_1!\alpha_2! \dots \alpha_e!$, is an isomorphism of R-modules.

Proof. The R-module structure of partial differentiation is defined by

$$\begin{gathered} \star : R \times S \to S \\ x^\alpha \star y^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{otherwise} \end{cases} \end{gathered}$$

That is

$$x^\alpha \star y^\beta = \frac{\beta_1!.\dots.\beta_e!}{(\beta_1-\alpha_1)!.\dots.(\beta_e-\alpha_e)!} y^{\beta-\alpha}.$$

We have

$$\phi(x^\alpha \circ y^\beta) = \phi(y^{\beta-\alpha}) = \frac{1}{(\beta-\alpha)!} y^{\beta-\alpha}$$

and

$$x^\alpha \star \phi(y^\beta) = x^\alpha \star \frac{1}{\beta!} y^\beta = \frac{1}{\beta!} . \frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha} = \frac{1}{(\beta-\alpha)!} y^{\beta-\alpha}.$$

Hence

$$\phi(x^\alpha \circ y^\beta) = x^\alpha \star \phi(y^\beta).$$

The inverse is defined by $\phi^{-1}(y^\alpha) = \alpha! y^\alpha$. □

Let $F = RF_1 + RF_2 + \cdots + RF_n$ be a finitely generated S-module, where each $F_i, 1 \leq i \leq n$ is a homogeneous polynomial of degree d. The homomorphism $C_F(u, v; e) : R_v \to S_u$ defined by

$$C_F(u, v; e)(r) = r \circ F,$$

where $u + v = d$, is called the *catalecticant homomorphism*, and the matrix associated to $C_F(u, v; e)$ with respect to the standard monomial bases elements of R_v and S_u (ordered lexicographically) is the *catalecticant matrix*, denoted by $Cat_F(u, v; e)$. The size of $\mathrm{Cat}_F(u, v; e)$ is $\binom{e+u-1}{u} \times \binom{e+v-1}{v}$.

Remark 3.1.4. When $\mathrm{char}(k) > 0$, there is an isomorphism between R-modules of divided power ring $\mathfrak{D}$ with contraction action and R-modules of S with partial differentiation. If $\mathrm{char}(k) = 0$, then $\phi : \mathfrak{D} \to S$ is an isomorphism, see [26].

Example 3.1.5.

Take $e = 3$ and $d = 3$. Let $c_{ijk} \in k$ for all $1 \leq i, j, k \leq 3$.

Let $F = c_{300}{y_1}^3 + c_{210}{y_1}^2y_2 + c_{201}{y_1}^2y_3 + c_{120}y_1{y_2}^2 + c_{102}y_1{y_3}^2 + c_{111}y_1y_2y_3 + c_{030}{y_2}^3 + c_{021}{y_2}^2y_3 + c_{012}y_2{y_3}^2 + c_{003}{y_3}^3$ and let $a = a_1x_1 + a_2x_2 + a_3x_3$. The catalecticant matrix associated to the catalecticant homomorphism $\mathrm{Cat}_{a \circ F}(1, 1; 3) : R_1 \to S_1$ is given by

$$\begin{pmatrix} a_1c_{300} + a_2c_{210} + a_3c_{201} & a_1c_{210} + a_2c_{120} + a_3c_{111} & a_1c_{201} + a_2c_{111} + a_3c_{102} \\ a_1c_{210} + a_2c_{120} + a_3c_{111} & a_1c_{120} + a_2c_{030} + a_3c_{021} & a_1c_{111} + a_2c_{021} + a_3c_{012} \\ a_1c_{201} + a_2c_{111} + a_3c_{102} & a_1c_{111} + a_2c_{021} + a_3c_{012} & a_1c_{102} + a_2c_{012} + a_3c_{003} \end{pmatrix}.$$

Lemma 3.1.6. *Suppose $S = k[y_1, y_2, \ldots, y_n]$ and $r \in R_i$ is fixed. Then the homomorphism $\Phi_r : S_j \to S_{j-i}$ defined by $\Phi_r(F) = r \circ F$ is a surjective homomorphism.*

Proof. If $r \in R_i$ and $F \in S_j$ then $\Phi_r(F) = r \circ F = 0 \Leftrightarrow S_{j-1} \circ r \circ F = 0 \Leftrightarrow S_{j-i} r \circ F = 0$. The kernel of Φ_r in S_j has vector space dimension $l(\ker \Phi_r) = l(S_j) - l(S_{j-i})$, so the image of Φ_r in S_{j-i} has dimension $l(S_j) - l(\ker \Phi(r)) = l(S_{j-i})$. Thus, Φ_r is surjective.

□

Theorem 3.1.7 (Macaulay's Inverse System Theorem)**.** *Let $R = k[x_1, x_2, \ldots, x_e]$ and $S = k[y_1, y_2, \ldots, y_e]$ be polynomial rings. There is a one-to-one correspondence between finitely generated nonzero R-submodules F in S and ideals $I = \operatorname{ann}_R(F)$ in R such that R/I is a local Artinian ring.*

If $I \subset R$ is an ideal then we define $I^\perp = \{F \in S : I \circ F = 0\}$. Note that $I^\perp$ is an R-submodule of S. The corresponding given as follows: The R-submodules $I^\perp$ of S are called Macaulay's Inverse System of I and

$(R/I)^\vee = \operatorname{Hom}_R(R/I, S) \cong I^\perp$. By Matlis duality

$$I^\perp \text{ is finitely generated} \iff A = R/I \text{ is Artinian}$$

If the dual module $I^\perp$ is cyclic, then

$$A \text{ is an Artinian Gorenstein ring} \iff I^\perp \cong A.$$

We have the following theorem from Iarrobino and Kanev (1999) which is a special case of this theorem.

Theorem 3.1.8 ([26])**.** *Let $A = R/I$ be an Artinian graded ring. Then A is Gorenstein of socle degree d if and only if $I = \operatorname{ann}_R(F) = \{r \in R | r \circ F = 0\}$, for some homogeneous polynomial F of degree d in $S = k[y_1, y_2, \ldots y_e]$.*

In other words, all Gorenstein quotients R/I arise in this fashion.

Example 3.1.9.

Let $e = 3$, $R = k[x_1, x_2, x_3]$ and $S = k[y_1, y_2, y_3]$. Considering the submodule $F = (y_1^2 + y_2^2, 2y_2^2 + y_3^2)$ in S, we can compute $\operatorname{ann}_R(F)$ using the contraction action ($\circ$). One finds out that $\operatorname{ann}_R(F) = (x_1x_3, x_2x_3, x_1x_2, x_1^2 - x_2^2 + 2x_3^2)$ and the corresponding Artinian ring $A = R/\operatorname{ann}_R(F)$ has Hilbert series $1 + 3t + 2t^2$ and k basis $\{1, x_1, x_2, x_3, x_2^2, x_3^2\}$.

Example 3.1.10.

Take $R = k[x_1, x_2]$ and $S = k[y_1, y_2]$ and choose $F = y_1^3 + y_2^3$. Then $I = \operatorname{ann}_R(F) = (x_1x_2, x_1^3 - x_2^3)$, and the quotient ring $A = k[x_1, x_2]/(x_1x_2, x_1^3 - x_2^3)$ has a k-basis $\{x_1, x_2, x_1^2, y_1^2, x_1^3\}$ with socle kx_1^3. Hence it is a Gorenstein ring of socle degree 3.

We prove that for the Gorenstein case there is a bijection between the R-submodule F in S and the quotient ring A/I.

Proposition 3.1.11. *Let $A = R/I$ be a Gorenstein ring with $I = \operatorname{ann}_R(F)$, for some homogeneous degree d element F in $S = k[y_1, y_2, \ldots, y_n]$.*
Define $M(F) = \{y \in S : y = r \circ F\}$ for some $r \in R$. Then there exists a bijection between R/I and $M(F)$.

Proof. Define the map $\phi : R \to M(F)$ by $\phi(r) = r \circ F$. Then ϕ is a homomorphism since, for all $r_1, r_2 \in R$ and $a \in M(F)$,

1. $\phi(r_1 + r_2) = (r_1 + r_2) \circ f = r_1 \circ F + r_2 \circ F = \phi(r_1) + \phi(r_2)$ for all $r_1, r_2 \in R$, and
2. $\phi(ar) = (ar) \circ F = a(r \circ F) = a\phi(r_2)$.

For any $y \in M(F)$, $y = r \circ F$, and there exist $r \in R$ such that $\phi(r) = r \circ F$. Thus, ϕ is a surjective homomorphism. By the First Isomorphism Theorem, ϕ induces an isomorphism $\phi' : R/\ker\phi \to M(F)$. We need to prove that $\ker\phi = \operatorname{ann}_R(F)$. Let $r \in \ker\phi$. Then $\phi(r) = 0$ implies $r \circ F = 0$, and $r \in \operatorname{ann}_R(F)$. That is, $\ker\phi \subseteq \operatorname{ann}_R(F)$. Now let $r \in \operatorname{ann}_R(F)$, which means $r \circ F = 0$. But $r \circ F \in M(F)$, so there exists an $\bar{r} \in R/\ker(\phi)$ such that $\phi'(\bar{r}) = r \circ F = 0$. Thus, $\bar{r} \in \ker\phi'$. Whence, $\ker\phi' = \{\bar{r} | r \in \ker\phi\}$, and $\phi(r) = 0$, and hence $r \in \ker(\phi)$. □

Fact 3.1.12. Let F be a homogeneous of degree d polynomial, and let $A = R/\operatorname{ann}(F)$. Then, A is an Artinian Gorenstein graded ring, and the socle of A is determined by any of the terms of F, see [26].

When A is an Artinian ring, a nonzero element $a \in A$ is an exact zero divisor if and only if the ideal $\operatorname{ann}_A(a)$ is principal:

Theorem 3.1.13. [20] *If A is an Artinian ring, then $\operatorname{ann}_A(a) = (b)$ implies $\operatorname{ann}_A(b) = (a)$.*

Proof. Let $\operatorname{ann}_A(a) = (b)$. Since $ab = 0$, we have $(a) \subseteq \operatorname{ann}_A(b)$. By length count and the isomorphisms $R/\operatorname{ann}(b) \cong (a)$ and $R/\operatorname{ann}(a) \cong (b)$, gives

$$l(A) = l(A/(b)) + l(b) = l(A/\operatorname{ann}(a)) + l(b) = l(a) + l(A/(a))$$

$$\leq l(\operatorname{ann}_A(b)) + l(A/\operatorname{ann}_A(b)) = l(A).$$

Therefore, $l(a) = l(\operatorname{ann}_A(b))$; hence $\operatorname{ann}_A(b) = (a)$. □

Chapter 4

EXISTENCE OF EXACT PAIRS OF ZERO DIVISORS

In this Chapter, all rings A are of the form $A = R/I$ for $R = k[x_1, \ldots, x_e]$. In Section 1, we discuss some known results about the existence of exact pairs of zero divisors in these rings of socle degree 2 and 3. We investigate some conditions when these rings of socle degree $d > 3$ contain linear pairs of exact zero divisors. In Section 3, we establish the main result, Theorem 4.3.2.

Throughout this section, d will denote the degree of the socle of A, and e will denote the embedding dimension of A.

4.1 Artinian Gorenstein rings of socle degree $d \leq 2$

The ring $A = k[x]/(x^2)$ is the only Artinian Gorenstein ring of socle degree 1, and we can easily see that it has (x, x) as an exact pair of zero divisors. For $e \geq 2$, all the Artinian rings of socle degree 1 are non-Gorenstein. Since these are the rings satisfying $\mathfrak{m}^2 = 0$, every $a \in A_1$ annihilates all the linear terms in A. Therefore, there is no exact pair of zero divisors.

Now consider the case $d = 2$ case. The ring $A = k[x]/(x^3)$ is the only Artinian Gorenstein ring of socle degree 2 when $e = 1$. While there is the exact pair of zero divisors (x, x^2), there is no linear pair of exact zero divisors.

For $e = 2$, every Artinian Gorenstein ring of socle degree $d = 2$ has an exact pair of zero divisors.

Proposition 4.1.1. *A Gorenstein local ring $(A, \mathfrak{m}, k)$ satisfying $\mathfrak{m}^3 = 0$ and $e = 2$ always admits an exact pair of zero divisors.*

Proof. Let $R = k[x_1, x_2]$ and $S = k[y_1, y_2]$ be polynomial rings as described in Chapter 3. Define a map $C_{\widetilde{a}\circ\widetilde{F}}(1, 1; 2) : R_1 \to S_1^{\#}$ by $C_{\widetilde{a}\circ\widetilde{F}}(0, 1; 2)(r) = r \circ \widetilde{a} \circ \widetilde{F} = r\widetilde{a} \circ \widetilde{F}$, where $\widetilde{a} = a_1x_1 + a_2x_2 \in A_1$ and $\widetilde{F} = c_1y_1^2 + c_2y_1y_2 + c_3y_2^2$. Then we have

$$\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(0, 1; 2) = \begin{pmatrix} a_1c_1 + a_2c_2 & a_1c_2 + a_2c_3 \end{pmatrix}.$$

Let b be in the kernel of the matrix. We can see that $\widetilde{b} = -(a_1c_2 + a_2c_3)x_1 + (a_1c_1 + a_2c_2)x_2$. If there is another linear element $c \in R_1$ not generated by $\widetilde{b}$ such that $ac = 0$, then the map $C_{a\circ F}(1, 1; 2)$ will be the zero map. We know that $\widetilde{a}$ is in $\mathrm{ann}_R(b)$. If there is another linear element $c \in R$, not generated by a such that $bc = 0$, then b would be a socle element, which contradicts our assumption that A is Gorenstein. This implies that, for every linear form $\widetilde{a}$ in A, there is exactly one linear form $b \in A_1$ such that $\widetilde{a}b = 0$. Next, we observe that there are no quadratic terms in $\mathrm{ann}_R(F)$ annihilating $\widetilde{a}$, otherwise we will have $Ab \neq A_2$, which is a contradiction again. Hence, $(\widetilde{a}, b)$ make an exact pair of zero divisors. □

We have shown that when A has socle degree 2 and if the embedding dimension $e = 2$, there exists a pair of exact zero divisors. But this is not the case when $e \geq 3$.

Proposition 4.1.2. *A Gorenstein local ring $(A, \mathfrak{m}, k)$ satisfying $\mathfrak{m}^3 = 0$ and $e \geq 3$ does not have a linear pair of exact zero divisors.*

Proof. Consider $k^{\#} = k[a_1, \ldots, a_e, c_\alpha : |\alpha| = 2]$ and $S^{\#} = k^{\#}[y_1, \ldots, y_e]$. Let $\widetilde{a} = a_1x_1 + a_2x_2 + a_3 + \cdots + a_e \in A_1$ and let $\widetilde{F} = \sum_{|\alpha|=2} c_\alpha y^\alpha$. The catalecticant matrix $\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(0, 1; e)$ is of size $1 \times e$. Since $e \geq 3$, this matrix has at least a two-dimensional kernel, which is equivalent to saying that, for every linear form $\widetilde{a}$ in R, there are at least two elements in the $\mathrm{ann}_R(F)$ annihilating $\widetilde{a}$. □

4.2 Artinian Gorenstein rings of socle degree $d = 3$

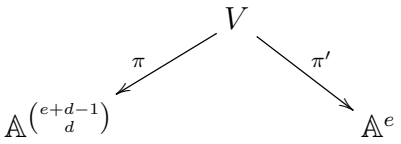
Conca, Rossi, and Valla [15] showed that generic Artinian Gorenstein standard graded algebras of socle degree 3 admit pairs of exact zero divisors. In this section, we give an alternate proof.

First we discuss elimination theory, which we will be using in the proof of our main theorem.

Definition 4.2.1. Given $I = (f_1, \ldots, f_e) \subset R$ the *j-th elimination ideal* I_j is the ideal of $k[x_{j+1}, \ldots, x_e]$ defined by

$$I_j = I \cap k[x_{j+1}, \ldots, x_e].$$

It is an easy exercise to check that the I_j are indeed ideals of $k[x_{j+1}, \ldots, x_e]$ for all j. We define the 0-th elimination ideal to be I itself; i.e., $I_0 = I$. This elimination corresponds to projecting a variety onto subvaritey of a lower dimensional subspace. Let I be an ideal generated by the determinant of $\mathrm{Cat}_{a \circ F}(1, 1; e)$ in $k[a_1, a_2, \ldots, a_e, c_\alpha : |\alpha| = 3]$, and let $V \subseteq \mathbb{A}^{\binom{e+d-1}{d}+e}$ be the variety $V(I)$. Suppose π and π' are the canonical projections from V on to $\mathbb{A}^{\binom{e+d-1}{d}}$ and $\mathbb{A}^e$ respectively.

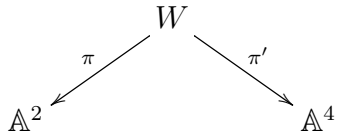


To see how this works, consider the following example.

Take $e = 2$, $d = 3$ and $\widetilde{F} = c_1 y_1^3 + c_2 y_1^2 y_2 + c_3 y_1 y_2^2 + c_4 y_2^3 \in k^{\#}[y_1, y_2] = S^{\#}$ where $k^{\#} = k[a_1, a_2, c_1, c_2, c_3, c_4]$. Then the catalecticant matrix corresponding to $C_{\widetilde{a} \circ \widetilde{F}}(1, 1; 2) R_1 \to S_1^{\#}$ for $\widetilde{a} = a_1 x_1 + a_2 x_2$ is

$$\mathrm{Cat}_{\widetilde{a} \circ \widetilde{F}}(1, 1; 2) = \begin{pmatrix} a_1 c_1 + a_2 c_2 & a_1 c_2 + a_2 c_3 \\ a_1 c_2 + a_2 c_3 & a_1 c_3 + a_2 c_4 \end{pmatrix}.$$

Let I be the ideal generated by the determinant of the above matrix and W be the variety $W(I)$. Then $I = (-a_1^2c_2 + a_1^2c_1c_3 - a_1a_2c_2c_3 - a_2^2c_3^2 + a_1a_2c_1c_4 + a_2^2c_2c_4)$. One can see that for any choices of a's, we get an equation in c's and the equation has a solution. Similarly, for any choices of c's, we get an equation in a's and there is a solution. This implies that for any choices of c's there are choices of a's which makes the determinant 0.



Let I_1 be the eliminating ideal defined by $I \cap k[c_1, c_2, c_3]$ and I_2 be the eliminating ideal defined by $I \cap k[a_1, a_2]$. We can use Macaulay 2 to compute these eliminating ideals and the result agrees that both I_1 and I_2 are the zero ideals. This would mean that for every choice of $c's$, there is $a \in R$ such that the determinant of the matrix is zero. We now want to make precise the term generic.

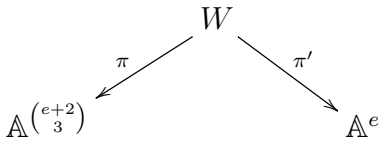
Definition 4.2.2. Suppose that objects correspond to points in some affine variety. A condition $\mathcal{C}$ on the objects is called an *open condition* if the points corresponding to those objects satisfying the condition $\mathcal{C}$ form a Zariski open subset of the variety. Objects satisfying an open condition are called *generic*.

Now we present the reformulated statement of the theorem of Conca, Rossi and Valla, and give a proof using our methods.

Theorem 4.2.3 ([15])**.** *A generic Artinian Gorenstein ring A of socle degree $d = 3$ admits a linear exact pair of zero divisors.*

Proof. Let $k^{\#} = k[a_1, \ldots, a_e, c_\alpha : |\alpha| = 3]$ and $S^{\#} = k^{\#}[y_1, \ldots, y_e]$. Suppose that $\widetilde{F} = \sum_{|\alpha|=3} c_\alpha y^\alpha$ is a homogeneous degree 3 polynomial in $S^{\#}$, and $\widetilde{a} = a_1x_1 + a_2x_2 + \cdots + a_ex_e$ is a nonzero linear form in $k^{\#}[x_1, \ldots, x_e]$.

We consider catalecticant homomorphisms from R to $S^{\#}$. The catalecticant matrix $\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(1,1;e)$ is an $e\times e$ matrix with entries of the form $\sum a_i c_\alpha$. Let J be the ideal in $\mathfrak{R}=k[a_1,\dots,a_e,c_\alpha:|\alpha|=3]$ generated by the determinant of the matrix $\mathrm{Cat}_{a\circ F}(1,1;e)$ and W be the variety $W(J)$ in $\mathbb{A}^{\binom{e+2}{3}+e}$. Suppose π and π' are the projections from W on to $\mathbb{A}^{\binom{e+2}{3}}$ and $\mathbb{A}^e$ respectively.



Since the variety W is defined by the $e\times e$ determinant, for any choices of $a\in\mathbb{A}^e$, we get an equation in c's and the equation has a solution. Similarly, for any choices of $c_j\in\mathbb{A}^{\binom{e+2}{3}}$, we get an equation in a's and there is a solution. This implies that for any choices of c's there are choices of a's which makes the determinant 0. Hence images under π and π' of W becomes the whole space. This implies that $R_1\cap\mathrm{Ker}(\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(1,1;e))\neq 0$. Let $b\in R_1\cap\mathrm{ann}_A(\widetilde{a})$ be nonzero. Let I be the ideal in $R=k[a_1,\dots,a_e,c_\alpha:|\alpha|=3]/J$ generated by $e-1\times e-1$ minors of $\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(1,1;e)$ and V be the variety $V(I)$.

Define $\ker(\widetilde{a}\widetilde{F})_i := \ker\Big(\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(2-i,i;e)\Big)$ and $\ker(\widetilde{F})_i := \ker\Big(\mathrm{Cat}_{\widetilde{F}}(3-i,i;e)\Big)$ for $i=2,3$, where we regard these matrices as homomorphisms $\mathfrak{R}^{\binom{e+i-1}{i}}\to\mathfrak{R}^{\binom{e-i+1}{2-i}}$ and $\mathfrak{R}^{\binom{e+i-1}{i}}\to\mathfrak{R}^{\binom{e-i+2}{3-i}}$ respectively. Therefore $\ker(\widetilde{a}\widetilde{F})_i$ and $\ker(\widetilde{F})_i$ are submodules of $\mathfrak{R}^{\binom{e+i-1}{i}}$. Also we let $(Rb)_i$ denote the column space of the matrix representing the map $R_{i-1}\to R_i$ for $i=2,3$. Then it is easy to see that $(Rb)_i+(\ker\widetilde{F})_i\subseteq(\ker\widetilde{a}\widetilde{F})_i$.

Let $\{V_i\}$ be the set of varieties defined by

$$V_i=V\Big(\mathrm{ann}_{\mathfrak{R}}\Big((\ker\widetilde{a}\widetilde{F})_{i+1}/((Rb)_{i+1})+(\ker\widetilde{F})_{i+1}\Big)\Big),$$

for all $i=1,2$.

We have constructed the varieties such that if $p \in \mathbb{A}^{\binom{e+2}{3}+e}$ lies outside $V_1 \cup V_2$, then the corresponding values of a_i and c_α result in $(\ker aF)_i = (Rb)_i + (\ker F)_i$. Equivalently, there is a non-empty Zariski open subset U of $\mathbb{A}^{\binom{e+2}{3}+e}$ such that for all $F \in U$, A admits an exact pair of zero divisor. □

Finally, we need to check that the open sets we considered are not empty. Let $F = \sum_{j=2}^{e} y_1 {y_j}^2$. One can see that the corresponding Artinian Gorenstein ring $A = R/(x_1^2, x_i x_j, x_i^2 - x_j^2, 1 < i < j)$ admits an exact pair of zero divisors (x_1, x_1).

Example 4.2.4. Let $\widetilde{a} = a_1x_1 + a_2x_2 + a_3x_3 \in R^{\#}$ and choose $F = y_1^3 + y_2^3 + y_3^3$ in $S = [y_1, y_2, y_3]$. Then

$$\mathrm{Cat}_{\widetilde{a}\circ F}(1,1;3) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$

The determinant of the matrix is $a_1a_2a_3$. Choose $a_1 = 0$, then the kernel of the above matrix is generated by $b = x_1$. Compute

$$\mathrm{Cat}_{b\circ F}(1,1;3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

and one can see that $\ker(b) = (x_2, x_3)$. This example shows that not all Artinian Gorenstein rings of socle degree 3 admit an exact pair of zero divisors. In other words, the open set consisting of points lying outside the variety $V_1 \cup V_2$ is not the whole space.

4.3 Artinian Gorenstein rings of socle degree $d > 3$

In the socle degree 3 case, the images under π and π' of the variety defined by $\det(Cat_{a\circ F}(1,1;e))= 0$ turn out to be the whole space. Thus, there always exists at least one linear term annihilating a.

However, in socle degree 4 or higher, the images of π and π' are not the whole space. There are some Gorenstein rings of socle degree 4 or higher which do not have any linear pairs of exact zero divisors.

Example 4.3.1.

Take $A = k[x_1, x_2, x_3]/(x_2^2x_3, x_1x_2x_3, x_1^2x_3 - x_2x_3^2, x_2^3 - x_1x_3^2 - x_3^3, x_1^2x_2 - x_3^3, x_1^3)$ corresponding to $F = y_1^2y_2^2 + y_1y_2^3 + y_2y_3^3 + y_1^2y_3^2$, and let $\widetilde{a} = a_1x_1 + a_2x_2 + a_3x_3$ be in R_1. Then the catalecticant matrix is

$$\mathrm{Cat}_{\widetilde{a}\circ F}(2,1;3) = \begin{pmatrix} 0 & a_2 & a_3 \\ a_2 & a_1+a_2 & 0 \\ a_3 & 0 & a_1 \\ a_1+a_2 & a_1 & 0 \\ 0 & 0 & a_3 \\ a_1 & a_3 & a_2 \end{pmatrix}.$$

We compute all 3×3 minors of this matrix and the ideal generated by the 3×3 minors is $(-a_1a_2^2 - a_1a_3^2 - a_2a_3^2, -a_1^2a_3 - a_1a_2a_3 - a_2^2a_3, a_1^2a_2 + a_1a_2^2 + a_1a_3^2, a_1^3 + a_1^2a_2 + a_1a_2^2, -a_2^2a_3, -a_2a_3^2, -a_1a_3^2 - a_2a_3^2, -a_1a_2a_3 - a_2^2a_3, -a_1^2a_3 - a_1a_2a_3 - a_2^2a_3, -a_1^2a_3 - a_1a_2a_3 - a_2^2a_3, a_1a_3^2, -a_2^3 - a_1^2a_3 - a_1a_2a_3 + a_2a_3^2, a_1^2a_2 - a_2^2a_3 + a_3^3, a_1^3 + a_1^2a_2 - 2a_1a_2a_3 - a_2^2a_3, -a_1a_2^2 - a_2^3 - a_1^2a_3 + a_1a_3^2 + a_2a_3^2, -a_1^2a_2 - a_1a_2^2 - a_2^3, -a_1^3 + a_1^2a_3 + 2a_1a_2a_3, a_1a_2a_3, a_1^2a_3 + a_1a_2a_3 - a_2a_3^2, -a_3^3, a_1^2a_3 - a_1a_3^2 - a_2a_3^2)$. As we can see all the 3×3 minors vanish if and only if $a_1 = a_2 = a_3 = 0$. Therefore, $\mathrm{Cat}_{\widetilde{a}\circ F}(2,1;3)$ has a trivial kernel in A_1 for every non-zero linear term $a \in k^{\#}[x_1, \ldots, x_e]$.

Then, by using [30, Lemma 2.15], we obtain $I_1 = (\mathrm{ann}(R_3 \circ \widetilde{F})) \cap R_1 = \emptyset$ for any $a \in A_1$. Hence A doesn't contain any linear pairs of exact zero divisors. This is a Gorenstein ring with the Hilbert series $H_A(t) = 1 + 3t + 6t^2 + 3t^3 + t^4$. Hence by [28], it doesn't have linear exact pairs of zero divisors.

Theorem 4.3.2. *There exist algebraic varieties $W, V, V_k, k = 1, \ldots, d-1$ of $\mathbb{A}^{\binom{e+d-1}{d}+e}$ such that every point $p \in W \setminus V \cup V_1 \cup \cdots \cup V_{d-1}$ corresponds to an Artinian Gorenstein ring A of socle degree d and a linear exact pair of zero divisors in A.*

Proof. Suppose that $\widetilde{F} = \sum_{|\alpha|=d} c_\alpha y^\alpha$ is a homogeneous degree d element in $S^\#$ and $\widetilde{a} = a_1x_1 + a_2x_2 + \cdots + a_ex_e$ is a nonzero linear form in $R^\#$. Again, we consider catalecticant homomorphisms from R to $S^\#$.

Define the map $C_{\widetilde{a}\circ\widetilde{F}}(d-2,1;e) : R_1 \to S^\#_{d-2}$ by $C_{\widetilde{a}\circ\widetilde{F}}(r) = r \circ \widetilde{a} \circ \widetilde{F} = r\widetilde{a} \circ \widetilde{F}$. We get $\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(d-2,1;e)$ of size $\binom{e+d-3}{d-2} \times e$ with entries of the form $\sum a_ic_\alpha$. Let J be the variety defined by $e \times e$ minors of $\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(d-2,1;e)$ and W be the variety $W(J)$.

Let I be the ideal in $\mathfrak{R} = k[a_1, \ldots, a_e, c_\alpha : |\alpha| = d]/J$ generated by $e-1$ by $e-1$ minors of $\mathrm{Cat}_{a\circ F}(d-2,1;e)$ and V be the variety $V(I)$ in $\mathbb{A}^{\binom{e+d-1}{d}+e}$. We will use the notation $\ker(\widetilde{a}\widetilde{F})_i$ for $\ker\Big(\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(d-1+i,i;e)\Big)$ and $\ker(\widetilde{F})_i$ for $\ker\Big(\mathrm{Cat}_{\widetilde{F}}(d-i,i;e)\Big)$ for all $2 \le i \le d$, where we regard these matrices as homomorphisms $\mathfrak{R}^{\binom{e+i-1}{i}} \to \mathfrak{R}^{\binom{e+d-i-2}{d-i-1}}$ and $\mathfrak{R}^{\binom{e+i-1}{i}} \to \mathfrak{R}^{\binom{e+d-i-1}{d-i}}$, respectively. Therefore $\ker(\widetilde{a}\widetilde{F})_i$ and $\ker(\widetilde{F})_i$ are submodules of $\mathfrak{R}^{\binom{e+i-1}{i}}$. Let $b \in \ker\big(\mathrm{Cat}_{\widetilde{a}\circ\widetilde{F}}(d-2,1;e)\big)$ and let $(Rb)_i$ denote the column space of the matrix representing the map $R_{i-1} \to R_i^\#$ for all $2 \le i \le d-1$. Then it is easy to see that $(Rb)_i + (\ker\widetilde{F})_i \subseteq (\ker\widetilde{a}\widetilde{F})_i$.

Let $\{V_i\}$ be the set of varieties defined by

$$V_i = V\big(\operatorname{ann}_{\mathfrak{R}}\Big((\ker aF)_{i+1}/((Rb)_{i+1}) + (\ker F)_{i+1}\Big)\big),$$

for all $1 \le i \le d-1$.

We have constructed the varieties such that if $p \in \mathbb{A}^{\binom{e+d-1}{d}+e}$ lies outside $V_1 \cup V_2 \cup \ldots V_{d-1}$, then the corresponding values of a_i and c_α result in $(\ker aF)_i = (Rb)_i + (\ker F)_i$. The union of all the varieties is a closed set and its complement is an open

set. Hence any point $p \in W \setminus V \cup V_1 \cup \cdots \cup V_{d-1}$ corresponds to an Artinian Gorenstein ring of socle degree d and a linear exact pair of zero divisors (a, b). $\square$

The following example demonstrates that these open sets we are considering are not empty.

Example 4.3.3.

Take $e \geq 3$ and $d \geq 3$ and choose $F = y_1 y_2^{d-1} + y_1 y_3^{d-1} + \cdots + y_1 y_e^{d-1}$.

We will use $I_k = (\text{ann}(R_{d-k} \circ F) \cap R_k$ [[30], Lemma 2.15] to compute the annihilator of F. Let $I = \text{ann}_R(F)$. Then by a simple calculation, we find that $R_{d-k} \circ F$ consists only of monomials of the form $(y_i^k, y_1 y_i^{k-1}, 1 < i \leq e)$ for all $i = 1, \ldots, d-2$. It is easy to see that $I_1 = 0$ and $I_2 \supseteq (x_1^2, x_i x_j, i \neq 1, i < j)$. Notice that if $x = x_1^{\alpha_1} x_j^{\alpha_j}, 1 < j \leq e$, then $I_k = (x_1^k, \sum_{|\alpha|=k} x^\alpha)$, where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_e^{\alpha_e}, \alpha \neq 1$. Now, for the case $k = d-1$, we have $R_1 \circ f = (y_i^{d-1}, y_1 y_i^{d-2}, 2 < i \leq e)$ and $I_{d-1} \supseteq (y_i^{d-1} - y_e^{d-1}, 2 < i \leq e)$. Now to prove that these are the only elements in I, we will use a dimension argument as follows. Consider the map $\phi : R_k \to D_0$. defined by $\phi(r) = r \circ (r_{d-k} \circ F)$, where $r_{d-k} \in R_{d-k}$. Since this map is a surjective and the dimension of $\ker(\phi) = \binom{e+k-1}{k} - 2e + 2$, we have that $\text{ann}(R_{d-k} \circ F) \cap R_k$ has only $\binom{e+k-1}{k} - 2e + 2$ elements for all k, with the exception of $k = 1$ and $k = d-1$. Now, I_k for $2 < k < d-2$ is the ideal generated by the elements in I_2. Hence, the annihilator of F is $I = (I_k, I_{d-1}) = (x_1^2, x_i x_j, x_i^{d-1} - x_{i+1}^{d-1}, 1 < i < j)$. Thus, $A = R/I$ is an Artinian Gorenstein ring having (x_1, x_1) as an exact pair of zero divisors .

Note that A has the Hilbert series of the form $H_A(t) = 1 + et + \sum_{i=2}^{d-2}(2e - 2)t^i + et^{d-1} + t^d$.

Example 4.3.4.

Take $d = 4$ and $e = 3$, and let $F = y_1y_2^3 + y_1y_3^3$. The catalecticant matrix $\mathrm{Cat}_{a\circ F}(2, 1; 3)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \\ a_2 & a_1 & 0 \\ 0 & 0 & 0 \\ a_3 & 0 & a_1 \end{pmatrix}.$$

All 3×3 minors $(a_2^2a_3, a_2a_3^2, -a_1a_2^2, -a_1a_3^2)$ of the matrix vanish if and only if $a_2 = a_3 = 0$ or $a_1 = a_2 = 0$ or $a_1 = a_3 = 0$. Without loss of generality, assume that $a_2 = a_3 = 0$ and $a_1 = 1$. Then one can see, $b = x_1$ generates the kernel of the matrix. Compute

$$(Rb)_2 = (x_1^2, x_1x_2, x_1x_3),$$

$$(\ker(a \circ F))_2 = (x_1^2, x_1x_2, x_1x_3, x_2, x_3)$$

and

$$(\ker(F))_2 = (x_1^2, x_2x_3).$$

Therefore, we have $\ker(a \circ F)_2 = (Rb)_2 + (\ker(F))_2$. Similarly compute

$$(Rb)_3 = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_3^2, x_1x_2x_3),$$

$$(\ker(a \circ F))_3 = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_3^2, x_1x_2x_3, x_2^2x_3, x_2x_3^2)$$

and

$$(\ker(F))_3 = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2x_3, x_2^2x_3, x_2x_3^2)$$

Therefore, we have $\ker(a \circ F)_3 = (Rb)_3 + (\ker(F))_3$.

Chapter 5

EXISTENCE OF TOTALLY REFLEXIVE MODULES

This chapter is organized as follows. In Section 1, we give some background information related to totally reflexive modules, and we study their construction from a pair of exact zero divisors. In Section 2, we present our results about the existence of totally reflexive modules over certain Artinian non-Gorenstein rings in the absence of exact pairs of zero divisors.

5.1 Totally Reflexive Modules

In the late 1960s, Auslander and Bridger first introduced the idea of totally reflexive modules under the name of modules of G-dimension zero. It was Avramov and Martsinkovsky who first referred to them as a totally reflexive modules in 2002 [4]. These modules have been studied by many researchers, using several different terminologies, such as modules of G-dimension zero, maximal Cohen-Macaulay modules, and (finitely generated) Gorenstein-projective modules [18]. Totally reflexive modules are of interest since they are essential objects in Gorenstein relative homological algebra. Over a Gorenstein ring, the totally reflexive modules are exactly the maximal Cohen-Macaulay modules, but they are known to exist over any ring. This is because free modules are always totally reflexive modules. It is also known that if a local ring is not Gorenstein, then it either has infinitely many indecomposable pairwise non-isomorphic totally reflexive modules, or it has none other than the free modules.

The simplest types of non-trivial totally reflexive modules can be constructed by using an exact pair of zero divisors. The existence of an exact pair of zero divisors is equivalent to the existence of a free resolution of $A/(a)$ of the form

$$\mathbb{F}: \qquad \cdots \to R \xrightarrow{b} R \xrightarrow{a} R \xrightarrow{b} R \xrightarrow{a} 0.$$

This complex is totally acyclic since $\mathbb{F} \cong \mathrm{Hom}(\mathbb{F}, R)$ and therefore the modules $R/(a) \cong (b)$ and $R/(b) \cong (a)$ are totally reflexive modules. Hence, the existence of exact zero divisors implies the existence of totally reflexive modules.

Although most of the recent constructions of these modules in the literature start with a pair of exact zero divisors, see for example [22], these modules may exist in the absence of exact zero divisors, see [12]. Our interest in this chapter arises in their existence over non-Gorenstein local rings that do not have exact pairs of zero divisors. We are only aware of two examples of rings which admit non-free totally reflexive modules but do not have exact pair of zero divisors. The first example of such ring was found in [12] over a field of characteristic 2. In a recent study, Vraciu and Atkins [2] constructed an example of a ring of embedding dimension 8 over a field of arbitrary characteristic. The following facts are known about totally reflexive modules.

Fact 5.1.1.

1. All free modules are totally reflexive modules. We call a nonzero totally reflexive module trivial if it is free.
2. If M is totally reflexive, then so is the dual M^* of M.
3. Over a Gorenstein ring, totally reflexive modules are exactly the maximal Cohen Macaulay modules.
4. Over a local non-Gorenstein ring with $\mathfrak{m}^2 = 0$, there exist only trivial totally reflexive modules [4].

5. If M is totally reflexive module, then so is every syzygy of M.

Now we illustrate the fact 5.1.1(2). If M is a maximal Cohen Macaulay ring then M^* is Maximal Cohen Macaulay. Therefore, $\mathrm{Ext}_A(M^*, A) = 0$ for all $i > 0$. This means if

$$\mathbb{F} : \ldots \to F_1 \to F_0 \to M \to 0$$

is a free resolution of M and

$$\mathbb{G} : \ldots \to G_1 \to G_0 \to M \to 0$$

is a free resolution of M^*, then

$$\mathbb{G}^* : 0 \to M^{**} \to G_0^* \to G_1^* \to \cdots \to$$

is exact. Since $M \cong M^{**}$, we can splice $\mathbb{F}$ and $\mathbb{G}^*$ together and get a totally acyclic complex

$$\mathbb{F}|\mathbb{G}^* : \cdots \to F_2 \to F_1 \to F_0 \to G_0^* \to G_1^* \to \cdots .$$

Gorenstein rings are characterized by the fact that every finitely generated module has finite G-dimension [35].

Theorem 5.1.2. [11, Theorem 1.25]. *Let A be local ring. For every finitely generated A-module M of finite G-dimension, we have the equality*

$$\text{G-dim}_A(M) = \mathrm{depth}(A) - \mathrm{depth}_A(M).$$

The condition G-dim$(M) = 0$ for maximal Cohen Macaulay modules follows from the Theorem 5.1.2.

5.2 Totally Reflexive Modules Without Exact Pair of Zero Divisors

As was mentioned earlier in this chapter, we will only consider Artinian non-Gorenstein local rings $(A, \mathfrak{m}, k)$. Thus for every such $(A, \mathfrak{m}, k)$, there exists $n \in \mathbb{N}$ such that $\mathfrak{m}^n = 0 \neq \mathfrak{m}^{n-1}$. As just mentioned, Artinian non-Gorenstein rings with $\mathfrak{m}^2 = 0$ admit only trivial totally reflexive modules. Therefore, the rings we will focus on are these rings with the cube of the maximal ideal being zero.

If $e = 1$, then A is a hypersurface, which is a Gorenstein ring.

If $e = 2$, then A is a complete intersection ring, which is also Gorenstein.

The smallest e for which $\mathfrak{m}^3 = 0$ non-Gorenstein rings exist is $e = 3$. If $e = 3$, then A has a Conca generator by [14, proof of Theorem 1]. An element $a \in A$ is called Conca generator if $a^2 = 0$ and $aA_1 = A_2$. Note that Conca generators are special type of exact zero divisors, that is, we have Conca generators when $a = b$. In this case there always exists a non-free totally reflexive modules.

Therefore, in addition to $\mathfrak{m}^3 = 0$, we will consider rings with an embedding dimension $e \geq 3$. We now give some known necessary conditions for the existence of totally reflexive modules.

Theorem 5.2.1. [13, Theorem A] *Let $(A, \mathfrak{m}, k)$ be a local ring that is not Gorenstein and has $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. If $\mathfrak{F}$ is a non-zero minimal acyclic complex of finitely generated free A-modules, then the ring A has the following properties:*

(i) $\operatorname{Soc}(A) = \mathfrak{m}^2$, *and*

(ii) $\operatorname{edim}(A) = \dim(\operatorname{Soc}(A)) + 1$. *In particular,* $\operatorname{length}(R) = 2\operatorname{edim}(R)$.

Theorem 5.2.2. [37, Theorem 3.1] *Let $(A, \mathfrak{m}, k)$ be a non-Gorenstein local ring with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. If there exists a non-free totally reflexive A-module M, then the following conditions hold:*

(i) The Hilbert series of A is $1 + et + (e-1)t^2$,

(ii) A is a Koszul algebra, and

(iii) M has a free resolution of the form

$$\cdots \to A^n \to A^n \cdots \to A^n \to M \to 0.$$

In other words, the resolution of M is linear with constant betti numbers.

Theorem 5.2.3. [12, Cor 8.5] *Let k be an infinite field and let $e \geq 0$ be an integer. A generic standard graded k-algebra A with Hilbert series $A(t) = 1 + et + (e-1)t^2$ has an exact zero divisor.*

It is shown in [12] that for

$$A = k[x_1, x_2, x_3, x_4]/(x_1^2, x_1x_4, x_2^2, x_2x_4, x_3^2, x_3x_4, x_4^2 - x_1x_2 - x_1x_3),$$

there are no exact zero divisors in A when $\mathrm{char}(k) = 2$. Observe that A is a quadratic algebra with Hilbert series $1 + 4t + 3t^2$. It was also mentioned in [14, Example 12] that there are Conca generators in A. If k does not have characteristic 2 or 3, then the elements $x_1 + x_2 + 2x_3 - x_4$ and $3x_1 + x_2 - 2x_3 + 4x_4$ form an exact pair of zero divisors in A, see([12]). This was the example used to show that non-free totally reflexive modules may exist even in the absence of exact zero divisors, and that exact zero divisors may also exist in the absence of Conca generators. Later in this section, we use our approach of using catalecticant matrices to illustrate the example from [12].

In a recent study, Vraciu and Atkins constructed an interesting example of ring of embeding dimension 8. The ring

$$A = k[x_1, \ldots, x_4, y_1, \ldots, y_4]/((x_1, \ldots, x_4)^2 + (y_1, \ldots, y_4)^2 + I),$$

where $I = (x_1, x_2)(y_3, y_4) + (x_3, x_4)(y_1, y_2) + ((\sum_{i=1}^{4} x_i)(\sum_{j=1}^{4} y_j))$ does not have any exact zero divisors, but has non-free totally reflexive modules.

We now give a class of rings of embedding dimension 5 and higher that have no exact zero divisors, but each ring has non-free totally reflexive modules.

Theorem 5.2.4. *Suppose that* $F = (f_1, f_2, f_3, \{f_i\}_{i=4}^{e-1})$ *where*

$$f_1 = -y_1y_2 + y_1y_4, f_2 = -y_1y_2 + y_2y_3, f_3 = -y_1y_2 + y_3y_4, f_i = \{-y_1y_2 + y_{i+1}^2\}_{i=4}^{e-1}.$$

Then $A := R/\operatorname{ann}_R(F)$, *where* $\operatorname{ann}_R(F) = (x_1x_2, x_1x_4, x_2x_3, x_3x_4, x_i^2x_e, x_ix_e, i = 1, 2, \ldots e-1)$ *is an Artinian ring with* $\mathfrak{m}^3 = 0$, *and Hilbert series of* A *is* $1+et+(e-1)t^2$.

Proof. We first want to find the generating elements of $\operatorname{ann}(F)$. We look for generators in each degree. Define the map $C_F(1, 1; e) : R_1 \to (S_1)^{e-1}$ by

$$C_F(1, 1; e)(r) = r \circ F$$

.

Look at the catalecticant map $C_F(1, 1; e) : R_1 \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}} (S_1)^3$ for $e = 5$. We denote the matrix associated with this map by B. Then B is comprised of

$$B_1^5 = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow f_1$$

$$\mathrm{B}_2^5 = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow f_2$$

$$B_3^5 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow f_3$$

and $C_F(1,1;e):\ R_1 \xrightarrow{[f_4]} (S_1)^3$, with

$$C_1^5 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow f_4.$$

For $e = 5$, we therefore have $C_F(1,1;e):\ R_1 \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}} (S_1)^3$, and

$$\mathrm{Cat}_F(1,1;e) = \left(\begin{array}{c} B_1^5 \\ B_2^5 \\ B_3^5 \\ \hline C_1^5 \end{array}\right),$$

which is

$$B = \left(\begin{array}{ccccc} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

We can observe that the rank of this matrix is 5, which implies that the the kernel of the matrix is $\{0\}$. Hence, $\operatorname{ann}_R(F)$ does not contain any linear terms in the case $e = 5$.

Next, we will use induction on e. Suppose the rank of the catalecticant matrix is $e-1$ for $e-1$ variables. For $e - 1 \geq 5$, the catelecticant matrix

$$\operatorname{Cat}_F(1, 1; e-1) = \left(\begin{array}{c} B_1^{e-1} \\ B_2^{e-1} \\ B_3^{e-1} \\ \hline C_1^{e-1} \\ C_2^{e-1} \\ \vdots \\ C_{e-5}^{e-1} \end{array}\right)$$

has $(e-1)(e-2)$ rows and $(e-1)$ columns where

$$B_i^{e-1} = \left[\begin{array}{c|c} B_i^{e-2} & 0 \\ \hline 0 & 0 \end{array}\right]$$

for all $i = 1, 2, 3$,

$$C_j^{e-1} = \left[\begin{array}{c|c} C_j^{e-2} & 0 \\ \hline 0 & 0 \end{array}\right]$$

for $1 \leq j < e-5$, and

$$C_{e-5}^{e-1} = \left[\begin{array}{c|c} M^{e-6} & 0 \\ \hline 0 & 1 \end{array}\right],$$

$$M^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$M^k \left[\begin{array}{c|c} M^{k-1} & 0 \\ \hline 0 & 0 \end{array}\right].$$

For any e, the catalecticant matrix is

$$\mathrm{Cat}_F(1,1;e) = \left(\begin{array}{c} B_1^e \\ B_2^e \\ B_3^e \\ \hline C_1^e \\ C_2^e \\ C_3^e \\ \hline \vdots \\ \hline C_{e-4}^e \end{array}\right),$$

having $e(e-1)$ rows and e columns, where $B_i^e = \left[\begin{array}{c|c} B_i^{e-1} & 0 \\ \hline 0 & 0 \end{array}\right]$, $C_j^e = \left[\begin{array}{c|c} C_j^{e-1} & 0 \\ \hline 0 & 0 \end{array}\right]$, and $C_j^e = \left[\begin{array}{c|c} M^{e-5} & 0 \\ \hline 0 & 1 \end{array}\right]$.

We have $\mathrm{Cat}_F(1,1;e) = \left(\begin{array}{c} B_1^e \\ B_2^e \\ B_3^e \\ \hline C_1^e \\ C_2^e \\ C_3^e \\ \vdots \\ C_{e-4}^e \end{array}\right) = \left[\begin{array}{c|c} B_1^{e-1} & 0 \\ 0 & 0 \\ \hline B_2^{e-1} & 0 \\ 0 & 0 \\ \hline B_3^{e-1} & 0 \\ 0 & 0 \\ \hline C_1^{e-1} & 0 \\ 0 & 0 \\ \hline C_2^{e-1} & 0 \\ 0 & 0 \\ \hline C_3^{e-1} & 0 \\ 0 & 0 \\ \hline \vdots & \vdots \\ \hline C_{e-5}^{e-1} & 0 \\ 0 & 1 \end{array}\right]$.

Since the matrix $\left(\begin{array}{c} B_1^{e-1} \\ 0 \\ B_2^{e-1} \\ 0 \\ B_3^{e-1} \\ 0 \\ \hline C_1^{e-1} \\ 0 \\ C_2^{e-1} \\ 0 \\ C_3^{e-1} \\ 0 \\ \vdots \\ C_{e-5}^{e-1} \\ 0 \end{array}\right)$ has rank $e-1$, then the catelecticant matrix $\mathrm{Cat}_F(1,1;e)$ has rank e.

We next we look for quadratic elements in $\operatorname{ann}_R(F)$. Define $C_F(0,2;e) : R_2 \to (S_0)^{e-1}$ by $C_F(0,2;e)(r) = r \circ F$. The catalecticant matrix $\operatorname{Cat}_{a\circ F}(0,2;e)$ has size $e-1 \times \binom{e+1}{2}$. Since $x_i x_j \circ F = x_i x_j \circ f_k = c_k^{ij}$ for all $1 \le i < j \le 3$ and $1 \le k \le e-1$, this map is surjective, and the dimension of it's kernel is $\binom{e+1}{2} - (e-1) = \frac{e^2-e+2}{2}$. It is easy to check that $x_i^2 \circ F = 0$ for all $i < 5$, and that $x_i x_j \circ F = 0$, for all i, j except x_1x_2, x_1x_4, x_2x_3 and x_3x_4. Also $(x_1x_2 + x_1x_4 + x_2x_3 + x_3x_4 + x_5^2 + \ldots + x_e^2) \circ F = 0$. This implies there are $4 + (\binom{e}{2} - 4) + 1 = \frac{e^2-e+2}{2}$ elements in the annihilator of F. The corresponding Artinian ring $A = R/I$ has $\binom{e+1}{2} - \frac{e^2-e+2}{2} = e-1$ generators in degree 2. Choose $\{x_1x_2, x_1x_4, x_2x_3, x_3x_4, x_{i\,i=5}^{2e-1}\}$ as the basis of A_2. Note that all the elements in A_3 are generated by the elements $x_1^2, x_2^2, x_3^2, x_1x_3$, and $\dim_k(A_3) = 0$. Thus $I = \operatorname{ann}_R(F)$ is a homogeneous ideal of degree 2 . Hence the Hilbert series of A is $1 + et + (e-1)t^2$, which proves the theorem. □

Theorem 5.2.5. *The Artinian ring $A = R/\operatorname{ann}_R(F)$ constructed above does not have an exact pair of zero divisors, but A has totally reflexive modules that are presented by the matrices*

$$\Phi = \begin{pmatrix} -x_1 - x_4 - x_5 - \cdots - x_e & x_1 + x_2 + x_3 \\ x_2 + x_4 & x_1 - x_4 + x_5 + x_6 + \cdots + x_e \end{pmatrix}$$

and

$$\Psi = \begin{pmatrix} -x_1 + x_4 - x_5 - \cdots - x_e & x_1 + x_2 + x_3 \\ x_2 + x_4 & x_1 + x_4 + x_5 + x_6 + \cdots + x_e \end{pmatrix}.$$

We use the following lemma to aid in the proof of the above theorem.

Lemma 5.2.6. *Let Φ and Ψ be the matrices given above. Then*

(i) The dimensions of the image of Φ and Ψ are $2e$.

(ii) The dimensions of the image of Φ^T and Ψ^T are $2e$.

Proof. (i) The image of Φ is its column space. Fix the basis of A as $\{x_1x_2, x_1x_4, x_2x_3, x_3x_4, x_i^2\}$, where $5 \leq i \leq e-1$, as in the previous theorem. One can see that the column space of Φ is the number of linearly independent columns of the transpose of the following matrix

$$\begin{pmatrix} -x_1 - x_4 - x_5 - \cdots - x_e & x_2 + x_4 \\ x_1 + x_2 + x_3 & x_1 - x_4 + x_5 + x_6 + \cdots + x_e \\ -x_1x_4 & x_1x_2 + x_1x_4 \\ x_1x_2 & -x_1x_4 \\ -x_1x_2 & 0 \\ x_1x_2 + x_2x_3 & x_1x_2 \\ -x_3x_4 & x_2x_3 + x_3x_4 \\ x_2x_3 & -x_3x_4 \\ -x_1x_4 & 0 \\ x_1x_4 + x_3x + 4 & x_1x_4 \\ -x_5^2 & 0 \\ 0 & x_5^2 \\ -x_6^2 & 0 \\ 0 & x_6^2 \\ \vdots & \vdots \\ -x_e^2 & 0 \\ 0 & x_e^2 \end{pmatrix}.$$

Since there are $2(e-4) - 2 + 10 = 2e$ linearly independent rows, the row space is $2e$. Similarly, we compute the column space of Ψ. The column space of the matrix Ψ is the number of linearly independent columns of the transpose of the following matrix

$$\begin{pmatrix} -x_1 + x_4 - x_5 - x_6 - \cdots - x_e & x_2 + x_4 \\ x_1 + x_2 + x_3 & x_1 + x_5 + x_6 + \cdots + x_e \\ -x_1x_4 & x_1x_2 + x_1x_4 \\ x_1x_2 & x_1x_4 \\ -x_1x_2 & 0 \\ x_1x_2 + x_2x_3 & x_1x_2 \\ 0 & x_2x_3 \\ x_2x_3 & x_3x_4 \\ -x_1x_4 & 0 \\ x_1x_4 + x_3x + 4 & x_1x_4 \\ -x_5^2 & 0 \\ 0 & x_5^2 \\ -x_6^2 & 0 \\ 0 & x_6^2 \\ \vdots & \vdots \\ -x_e^2 & 0 \\ 0 & x_e^2 \end{pmatrix}.$$

One can show that these rows are linearly independent. Hence $\dim \mathrm{Im}(\Psi) = \dim \mathrm{Im}(\Psi) = 2e$.

(ii) The image of Φ^T is its column space. Fix the basis of A as $\{x_1x_2, x_1x_4, x_2x_3, x_3x_4, x_i^2\}$, where $5 \leq i \leq e-1$, as in the previous theorem. One can see that the column space

of Φ^T is the number of linearly independent columns of the transpose of the following matrix

$$\begin{pmatrix} -x_1 - x_4 - x_5 - \cdots - x_e & x_1 + x_2 + x_3 \\ x_2 + x_4 & x_1 - x_4 + x_5 + x_6 + \cdots + x_e \\ -x_1x_4 & x_1x_2 \\ x_1x_2 + x_1x_4 & -x_1x_4 \\ -x_1x_2 & -x_1x_2 + x_2x_3 \\ 0 & x_1x_2 \\ -x_3x_4 & x_2x_3 \\ x_2x_3 + x_3x_4 & 0 \\ -x_1x_4 & x_1x_4 + x_3x_4 \\ 0 & x_1x_4 \\ -x_5^2 & 0 \\ 0 & x_5^2 \\ -x_6^2 & 0 \\ 0 & x_6^2 \\ \vdots & \vdots \\ -x_e^2 & 0 \\ 0 & x_e^2 \end{pmatrix}.$$

Since there are $2(e-4) - 2 + 10 = 2e$ linearly independent rows, the row space is $2e$. Similarly, we compute the column space of Ψ^T. The column space of the matrix Ψ^T is the number of linearly independent columns of the transpose of the following matrix

$$\begin{pmatrix} -x_1+x_4-x_5-x_6-\cdots-x_e & x_2+x_4 \\ x_1+x_2+x_3 & x_1+x_5+x_6+\cdots+x_e \\ -x_1x_4 & x_1x_2 \\ x_1x_2+x_1x_4 & x_1x_4 \\ -x_1x_2 & x_1x_2 \\ -x_1x_2 & x_1x_2+x_2x_3 \\ 0 & x_1x_2 \\ x_3x_4 & x_2x_3 \\ x_2x_3+x_3x_4 & x_3x_4 \\ -x_1x_4 & x_1x_4+x_3x_4 \\ -x_5^2 & 0 \\ 0 & x_5^2 \\ -x_6^2 & 0 \\ 0 & x_6^2 \\ \vdots & \vdots \\ -x_e^2 & 0 \\ 0 & x_e^2 \end{pmatrix}.$$

□

Now we give the proof of the theorem 5.2.5

Proof. First we will show that there are not any exact zero divisors in A. Then we will prove that the modules presented by given matrices are totally reflexive modules over A.

Define the map $C_{a\circ F}(0,1;e):\ R_1 \to (S_0)^{e-1}$ by $C_{a\circ F}(0,1;e)(r) = r\circ a\circ F = ra\circ F$ for $a = a_1x_1 + \cdots + a_ex_e$. The catalecticant matrix $Cat_{a\circ F}(0,1;e)$ is the $e-1\times e$ matrix

$$Cat_{a\circ F}(0,1;e) = \begin{pmatrix} -a_2+a_4 & -a_1 & 0 & a_1 & 0 & 0 & \cdots & 0 \\ -a_2 & -a_1+a_3 & a_2 & 0 & 0 & 0 & \cdots & 0 \\ -a_2 & -a_1 & a_4 & a_3 & 0 & 0 & \cdots & 0 \\ -a_2 & -a_1 & 0 & 0 & a_5 & 0 & \cdots & 0 \\ -a_2 & -a_1 & 0 & 0 & 0 & a_6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_2 & -a_1 & 0 & 0 & 0 & 0 & \cdots & a_e \end{pmatrix}.$$

This matrix has at least a one dimensional kernel, and it is easy to see that $b = a_1x_1 - a_2x_2 + a_3x_3 - a_4x_4$ is in the kernel. That is, $b\in \operatorname{ann}_R(a\circ F)$. If there are other linear elements in the kernel then the matrix $Cat_{a\circ F}(0,1;e)$ does not have full rank. Then (a,b) will not be an exact pair of zero divisors.

Now, assume that $\operatorname{ann}_A(a)\cap A_1 = (b)$. This implies that the matrix $(Cat_a\circ f(2,1;3))$ has rank $e-1$. We repeat the process to find the catalecticant matrix $Cat_{b\circ F}(0,1;e)$, which is

$$Cat_{b\circ F}(0,1;e) = \begin{pmatrix} a_2-a_4 & -a_1 & 0 & a_1 & 0 & 0 & \cdots & 0 \\ a_2 & -a_1+a_3 & -a_2 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & -a_1 & -a_4 & a_3 & 0 & 0 & \cdots & 0 \\ a_2 & -a_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & -a_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_2 & -a_1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is easy to see that $\operatorname{ann}_R(b) = (a, x_5, x_6, \ldots x_e)$. Note that $x_i \neq a$, for all $i = 5,6,\ldots e$, otherwise the matrix $Cat_{a\circ F}(0,1;e)$ will not have full rank, which contradicts the assumption. We see that for any non-zero linear form a in A, there is another linear form $b\in A$ such that $\operatorname{ann}_R(a) = (b)$. But $\operatorname{ann}_R(b) = (a, x_5, x_6, \ldots x_e)$, which implies that A does not have any exact zero divisors.

Next, we want to show that the modules presented by the matrices Φ and Ψ are totally reflexive modules. It is easy to verify that the products $\Phi\Psi$ and $\Psi\Phi$ are zero, whence $\mathbb{F} = \cdots \xrightarrow{\Phi} A^2 \xrightarrow{\Psi} A^2 \xrightarrow{\Phi} A^2 \xrightarrow{\Psi} \cdots$ is a complex.

We shall first verify that $\mathbb{F}$ is totally acyclic. To see that, we must verify the equalities $\operatorname{Im}\Psi = \operatorname{Ker}\Phi$ and $\operatorname{Im}\Phi = \operatorname{Ker}\Psi$. By Lemma 5.2.3, we have $\dim(\operatorname{Im}(\Phi)) = \dim(\operatorname{Im}(\Psi)) = 2e$. But $\dim(\ker(\Phi)) = \dim(R^2) - \dim(\operatorname{Im}(\Phi)) = 4e - 2e = 2e$, and $\dim(\ker(\Psi)) = \dim(R^2) - \dim(\operatorname{Im}(\Psi)) = 4e - 2e = 2e$. Therefore the complex is exact, and hence $\mathbb{F}$ is acyclic. Since the maps in the dual complex $\operatorname{Hom}_A(\mathbb{F}, R)$ are the transpose matrices Φ^T and Ψ^T, its dual $\mathbb{F}^*$ is also a complex. Hence the complex is totally acyclic, and the module M, presented by Φ, is totally reflexive. Moreover, the first syzygy of M presented by Ψ is also a totally reflexive module . $\square$

We want to look at the example mentioned in [12]. This is the example from Conca in [14, Example 12] of a standard graded k-algebra with Hilbert series $1+4t+3t^2$ and $(0 : \mathfrak{m}) = \mathfrak{m}^2$ having no Conca generator of $\mathfrak{m}$.

We illustrate the example from [12] using our approach.

Example 5.2.7.

Let $e = 4$, $A = k[x_1, x_2, x_3, x_4]/(x_1^2, x_1x_4, x_2^2x_2x_4, x_3^2, x_3x_4, x_4^2 - x_1x_2 - x_1x_3)$ and the corresponding F is $(-y_1y_2 + y_1y_3, y_2y_3, -y_1y_2 + y_4^2)$ in S. Then

$$Cat_{a\circ F}(0, 1; 4) = \begin{pmatrix} -a_1 + a_3 & -a_1 & a_1 & 0 \\ 0 & a_3 & a_2 & 0 \\ -a_2 & a_1 & 0 & a_4 \end{pmatrix}$$

has

$$b = (a_1a_2a_4 + a_1a_3a_4)x_1 + (-a_1a_2a_4 + a_2a_3a_4)x_2 + (a_1^2a_2 - a_1a_2^2 - 2a_1a_2a_3)x_4$$

in its kernel and

$$\mathrm{Cat}_{b\circ F}(0,1;4) = \begin{pmatrix} a_2^2a_4 - a_3^2a_4 & a_1a_3a_4 + a_1a_2a_4 & -a_1a_2a_4 - a_1a_3a_4 & 0 \\ 0 & a_3^2a_4 - a_2a_3a_4 & a_2^2a_4 - a_2a_3a_4 & 0 \\ a_2^2a_4 - a_2a_3a_4 & -a_1a_2a_4 - a_1a_3 & 0 & 2a_1a_2a_3 \end{pmatrix}$$

In the matrix above, notice that if the characteristic of k is 2, then the last column is 0. Therefore, a and b are not exact pair of zero divisors in A.

Chapter 6

WEAK LEFSCHETZ PROPERTY

The Weak and Strong Lefschetz properties are strongly connected to many topics in algebraic geometry, commutative algebra and combinatorics. Some of these connections are quite surprising and are still not completely understood, and much work remains to be done. In this chapter, we give an overview of some known results on the Weak Lefschetz property and provide a connection between exact zero divisots and the Weak Lefschetz property for Artinian rings.

6.1 Some Known Results

Definition 6.1.1. Let a be a general linear form. We say that A has the *Weak Lefschetz Property (WLP)* if the homomorphism induced by multiplication by a, $A_i \xrightarrow{a} A_{i+1}$ has maximal rank for all i (i.e. is injective or surjective). We expect maximum rank $rk(a) = \min\{\dim A_i, \dim A_{i+1}\}$. The set of linear forms a for which the map a has maximum rank is an open set in R_1, hence the use of the term general linear form. We say that A has the *Strong Lefschetz Property (SLP)* if $A_i \xrightarrow{a^d} A_{i+d}$ has maximal rank for all i and d (i.e. is injective or surjective).

Theorem 6.1.2. [19] *Let $R = k[x, y, z]$, where $char(k) = 0$. Let $I = (F_1, F_2, F_3)$ be a complete intersection. Then R/I has the WLP.*

Proposition 6.1.3. [19, Proposition 4.4] *Every Artinian ideal in $k[x, y]$ with $char(k) = 0$ has the Strong Lefschetz property (and consequently also the Weak Lefschetz property).*

One of the most interesting open problems in this field is whether all embedding dimension 3 graded Artinian Gorenstein algebras have the WLP.

Example 6.1.4. [19, Example 4.3] Let R be the ring $k[u, v, x, y, z]$ and let $F = xu^2 + yuv + zv^2$. The Gorenstein ring $A = R/\operatorname{ann}(F)$ algebra has neither the Weak Lefschetz property nor the Strong Lefschetz property. However, now take the polynomial $g = uF$. Then the corresponding Gorenstein ring $A = R/\operatorname{ann}(g)$ has the Weak Lefschetz property but not the Strong Lefschetz. The Artinian algebra $A = k[x, y, z]/(x^3, y^3, z^3, (x + y + z)^3)$ has the WLP; A does NOT have the SLP because (a^3) fails to have maximal rank.

Theorem 6.1.5. [19] *If $char(k) = 0$ and I is any homogeneous ideal in $k[x, y]$ then R/I has the SLP.*

Theorem 6.1.6. *(Stanley (1980), J. Watanabe (1987), Reid-Roberts-Roitman (1991)) Let $R = k[x_1, \ldots, x_r]$, where k has characteristic zero. Let I be an Artinian monomial complete intersection, i.e. $I = (x_1^{a_1}, ..., x_r^{ar})$ Then R/I has the SLP. In particular, R/I has the WLP.*

Theorem 6.1.7. [2, 3.6] *Let I be homogeneous ideal in R defining $A = R/I$. Suppose that the ring A satisfies $\mathfrak{m}^3 = 0$ and has Hilbert series $1 + et + (e - 1)t^2$. If A admits an exact pair of zero divisors, then A has WLP.*

Proof. Consider a general linear form $a = a_1x_1 + \ldots a_ex_e$. The $e - 1 \times e$ matrix associated to the multiplication map $A_1 \xrightarrow{a} A_2$ having maximal rank is an open condition. If (a, b) is an exact pair of zero divisors then all $e - 1 \times e - 1$ minors are not identically zero. Therefore, the open condition is non-empty. □

Conjecture 6.1.8. *Suppose A is a quadratic Artinian Algebra and has Hilbert series $1 + et + (e - 1)t^2$. If A has WLP, then it admits an exact pair of zero divisors.*

If A is quadratic, then we believe that with all the hypotheses in the conjecture, A admits an exact pair of zero divisors. The example below shows that A has WLP but does not have an exact pair of zero divisors. This is the case when A is not quadratic.

Example 6.1.9. [2, Example 3.7] Let $A = k[x_1, x_2]/(x_1^2 - x_2^2, x_1^2 - x_1x_2, x_1^3)$. Then A is an Artinian ring with Hilbert Series $1 + et + (e-1)t^2$, but A is not quadratic. The ring A has WLP since the multiplication by $a_1x_1 + a_2x_2 : A_1 \to A_2$ is surjective if $a_1 + b_1 \neq 0$. Every linear term $a \in A$ has at least one linear term, say, b in the annihilator and also a quadratic annihilator which is not multiple of b. Hence A does not have an exact pair of zero divisors.

Theorem 6.1.10. [31, Theorem 5.11] *Let I be a homogeneous ideal in R defining a Gorenstein graded ring $A := R/I$ of dimension zero. Suppose that char$(k) = 0$ and the number of the minimal generators of* $\operatorname{ann}_R(a)$ *over $k[x_1, x_2, x_3]$ is less than or equal to two for a general linear form a. Then A has the WLP.*

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