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The direct and inverse scattering problems for the third-order operator

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THE DIRECT AND INVERSE SCATTERING PROBLEMS FOR THE THIRD-ORDER OPERATOR

by

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ABSTRACT

THE DIRECT AND INVERSE SCATTERING PROBLEMS
FOR THE THIRD-ORDER OPERATOR

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We consider the full-line direct and inverse scattering problems for the third-order ordinary differential equation containing two potentials decaying sufficiently fast at infinity. The direct scattering problem consists of the determination of the scattering data set when the two potentials are known. The scattering data set is made up of the corresponding scattering coefficients and the bound-state information. On the other hand, the inverse scattering problem involves the recovery of the two potentials when the scattering data set is available. We formulate the inverse scattering problem via a related Riemann–Hilbert problem on the complex plane. We describe the recovery of the two potentials from the solution to that Riemann–Hilbert problem. We also mention how the Riemann–Hilbert problem leads to a system of Marchenko integral equations. The recovery of the potentials from the solution to the Marchenko system will be published elsewhere.

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CHAPTER 1

INTRODUCTION

1.1 The goal of the thesis

Consider the third-order ODE (ordinary differential equation)

$$\psi''' + Q(x)\,\psi' + P(x)\psi = k^3\psi, \qquad x \in \mathbb{R}, \tag{1.1}$$

where x is the independent spacial variable, the prime denotes the x-derivative, k^3 is the spectral parameter, ψ is the dependent variable known as the wavefunction, and the coefficients $Q(x)$ and $P(x)$ are the potentials. Even though k^3 is the spectral parameter, we at times refer to k as the spectral parameter. In general, $Q(x)$ and $P(x)$ are complex-valued functions of x and we assume that they belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Most of our results hold under weaker conditions on the potentials, but for simplicity and clarity, we assume that $Q(x)$ and $P(x)$ belong to the Schwartz class. We recall the definition of the Schwartz class below. Note that for any nonnegative integer n, we use $f^{(n)}(x)$ for the nth derivative of a function $f(x)$, with the understanding that $f^{(0)}(x)$ corresponds to $f(x)$ itself.

Definition 1.1.1. We say that a function $f(x)$ belongs to $\mathcal{S}(\mathbb{R})$ if $f \in \mathcal{C}^\infty(\mathbb{R})$, i.e. infinitely differentiable, and for every pair of nonnegative integers m and n, the quantity $x^m f^{(n)}(x)$ tends to 0 as $x \to \pm\infty$.

Our goal in the thesis is to study the direct and inverse scattering problems for (1.1). In particular, our thesis develops a viable method to solve the corresponding inverse scattering problem. This is done by formulating a relevant Riemann–Hilbert problem whose solution yields the solution to the inverse scattering problem for (1.1). We also mention how that Riemann–Hilbert problem leads to a system of Marchenko integral equations. The details related to the system of Marchenko integral equations and the solution of the inverse scattering by using the solution to the Marchenko system will be published elsewhere.

Without loss of generality, the nth-order linear ODE with $n \geq 2$ given by

$$\psi^{(n)} + a_{n-1}(x)\,\psi^{(n-1)} + a_{n-2}(x)\,\psi^{(n-2)} + \cdots + a_1(x)\,\psi' + a_0(x)\,\psi = k^n \psi, \qquad x \in \mathbb{R}, \quad (1.2)$$

can be transformed into

$$\tilde{\psi}^{(n)} + \tilde{a}_{n-2}(x)\,\tilde{\psi}^{(n-2)} + \cdots + \tilde{a}_1(x)\,\tilde{\psi}' + \tilde{a}_0(x)\,\tilde{\psi} = k^n \tilde{\psi}, \qquad x \in \mathbb{R}, \quad (1.3)$$

where the coefficient of $\tilde{\psi}^{(n-1)}$ is zero. This can be achieved by relating ψ to $\tilde{\psi}$ via

$$\psi(x) = \Gamma(x)\,\tilde{\psi}(x), \quad (1.4)$$

and by choosing $\Gamma(x)$ as

$$\Gamma(x) = \Gamma(x_1)\,\exp\left(-\frac{1}{n}\int_{x_1}^{x} dy\, a_{n-1}(y)\right), \quad (1.5)$$

where x_1 is an arbitrarily chosen fixed point in the x-domain of the wavefunction ψ. Hence, it is not surprising that the second derivative ψ'' is missing in (1.1).

Since we can interpret each of the coefficients $\tilde{a}_j(x)$ for $0 \leq j \leq n-2$ as a potential, we see that the linear ODE (1.3) contains $(n-1)$ potentials making up the set $\{\tilde{a}_j(x)\}_{j=0}^{n-2}$. For example, the third-order ODE (1.1), where $n = 3$, has the set of two potentials, given by $\{Q(x),\, P(x)\}$.

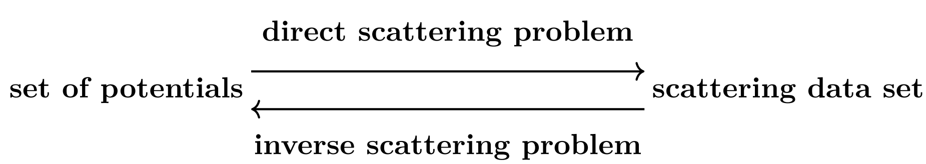

Figure 1.1: The direct and inverse scattering problems for an nth-order ODE.

The direct problem for (1.3) involves determining the scattering data set for (1.3) from the set $\{\tilde{a}_j(x)\}_{j=0}^{n-2}$ consisting of $(n-1)$ potentials. Meanwhile, the inverse scattering problem for (1.3) consists of the determination of those $(n-1)$ potentials when the scattering data set is known. Figure 1.1 helps illustrate the direct and inverse scattering problems for the nth-order ODE given in (1.3). In particular, the direct scattering problem for the third-order ODE (1.1) corresponds to the determination of the scattering data set for (1.1) when the potential set $\{Q(x),\, P(x)\}$ is given. On the other hand, the inverse scattering problem for (1.1) consists of the determination of the potentials $Q(x)$ and $P(x)$ when the corresponding scattering data set is specified.

The scattering data set comprises the scattering coefficients and the bound-state information. The scattering coefficients are certain functions of the spectral parameter k, and they are obtained from the spacial asymptotics of certain particular solutions to (1.1). The bound states for (1.1) consists of nontrivial solutions to (1.1) which are square integrable in $x \in \mathbb{R}$. In fact, when the potentials $Q(x)$ and $P(x)$ belong to the Schwartz class, the bound-state solutions must decay exponentially as $x \to \pm\infty$. The bound-state solutions occur at certain k-values in the complex k-plane at which the corresponding particular solutions become linearly dependent.

1.2 The comparison with the second-order case

Consider the analog of (1.1) in the second-order case, namely, consider the ODE

$$-\psi'' + V(x)\,\psi = k^2\psi, \qquad x \in \mathbb{R}, \tag{1.6}$$

where k^2 is the spectral parameter and $V(x)$ is the potential. The direct and inverse scattering problems for (1.6) are well understood [3,8,13,16,19,30,31,35] when the potential $V(x)$ is real valued and belongs to the Faddeev class. The Faddeev class consists of potentials $V(x)$ satisfying the integrability condition

$$\int_{-\infty}^{\infty} dx\,(1+|x|)\,|V(x)| < +\infty, \tag{1.7}$$

where we use the vertical bars to denote the absolute value. The second-order linear ODE (1.6) is known [21,28,33,34,37] as the Schrödinger equation, and it describes the quantum mechanical behavior of a particle under the influence of the potential $V(x)$.

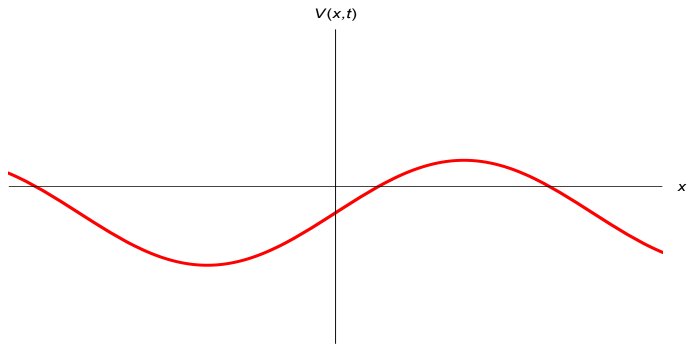

Figure 1.2: The snapshot at time t of $V(x,t)$ depicting the height of water from equilibrium.

If the potential $V(x)$ appearing in (1.6) also depends on the time parameter t, then the corresponding Schrödinger equation is given by

$$-\psi'' + V(x,t)\,\psi = k^2\psi, \qquad x \in \mathbb{R}, \tag{1.8}$$

where $V(x,t)$ is the time-dependent potential. The linear second-order ODE in (1.8) is related to the NPDE (nonlinear partial differential equation)

$$V_t(x,t) - 6V(x,t)\,V_x(x,t) + V_{xxx}(x,t) = 0, \tag{1.9}$$

where the subscripts denote the appropriate partial derivatives. The NPDE given in (1.9) is known [25] as the KdV (Korteweg–de Vries) equation, and it describes [1, 2, 17, 18, 25, 27, 36, 38] the propagation of surface water waves along narrow, shallow canals. Figure 1.2 depicts the quantity $V(x,t)$, which has the physical interpretation as the height of water from the equilibrium at time t and at location x along the canal.

We can write (1.6) as

$$L\psi = k^2\psi, \tag{1.10}$$

where L is the second-order linear differential operator given as

$$L = -D^2 + V(x), \tag{1.11}$$

with $D := d/dx$ and $D^2 := d^2/dx^2$. If the potential $V(x)$ is real valued and belongs to the Faddeev class specified by (1.7), then L is a selfadjoint differential operator. As a result of the selfadjointness of the corresponding differential operator, the direct and inverse scattering problems for (1.6) are well understood [1, 2, 17, 18, 25, 27, 36, 38].

In a manner analogous to (1.6), we can write (1.1) as $L\psi = k^3\psi$, where L is the third-order linear differential operator defined as

$$L = D^3 + Q(x)D + P(x), \tag{1.12}$$

with $D^3 := d^3/dx^3$. In general, the linear differential operator in (1.12) is not selfadjoint, even when both $Q(x)$ and $P(x)$ are real valued. Hence, (1.1) is in general not associated with a selfadjoint operator. Consequently, the analysis of the direct and inverse scattering problems for (1.1) are more challenging.

1.3 The nonlinear system associated with the third-order ODE

If the potentials $Q(x)$ and $P(x)$ appearing in (1.1) also depend on the time parameter t, then the time-evolved analog of (1.1) is given by

$$\psi''' + Q(x,t)\,\psi' + P(x,t)\psi = k^3\psi, \qquad x \in \mathbb{R}, \tag{1.13}$$

where $Q(x,t)$ and $P(x,t)$ are the time-evolved potentials. The time-evolved third-order equation (1.13) is associated with the nonlinear system of two equations given by

$$\begin{cases} Q_t + Q_{xxxxx} + 5Q_x\,Q_{xx} + 5Q\,Q_{xxx} + 5Q^2\,Q_x + 15Q_{xx}\,P + 15Q_x\,P_x - 30PP_x = 0, \\ P_t + P_{xxxxx} + 5Q\,P_{xxx} + 15Q_x\,P_{xx} + 20Q_{xx}\,P_x + 5Q^2\,P_x + 10Q_{xxx}\,P \\ \qquad\qquad\qquad - 15P\,P_{xx} + 10Q\,Q_x\,P - 15P_x^2 = 0, \end{cases} \tag{1.14}$$

where we have suppressed the arguments in $Q(x,t)$ and $P(x,t)$ for simplicity. The nonlinear system in (1.14) can be derived by using the pair of linear differential operators L and A given by

$$\begin{cases} L = D^3 + QD + P, \\ A = 9D^5 + 15QD^3 + (15P + 15Q_x)D^2 + (10Q_{xx} + 15P_x + 5Q^2)D + (10P_{xx} + 10QP), \end{cases} \tag{1.15}$$

where it is clear that $D^5 := d^5/dx^5$. In the terminology of the field of integrable evolution equations [1, 2, 4–7, 17, 18, 27, 29, 36, 38], the linear differential operators L and A in (1.15) form the Lax pair (L, A) for the system of NPDEs in (1.14). In other words, the operator equation

$$L_t + LA - AL = 0, \tag{1.16}$$

yields the nonlinear system (1.14). This means that the combined differential operator $L_t + LA - AL$ appearing on the left hand side in (1.16) is actually the zero multiplication operator. From the first line of (1.15), we have

$$L_t = Q_t\, D + P_t, \tag{1.17}$$

and we obtain the differential operator LA by using the operator multiplication of the operators L and A in that order, and similarly we obtain the differential operator AL by multiplying the operators A and L in that order.

In an analogous manner, the KdV equation (1.9) is obtained by using the operator equation (1.16) with the help of the Lax pair (L, A), where we now have

$$\begin{cases} L = -D^2 + V, \\ A = -4D^3 + 6VD + 3V_x, \end{cases} \tag{1.18}$$

with the understanding that V represents $V(x, t)$.

In the two special cases with $P(x, t) \equiv 0$ and $P(x, t) \equiv Q_x(x, t)$, respectively, the nonlinear system (1.14) reduces to the single NPDE given by

$$Q_t + Q_{xxxxx} + 5Q_x\, Q_{xx} + 5Q\, Q_{xxx} + 5Q^2\, Q_x = 0. \tag{1.19}$$

In other words, the last three terms on the left-hand side of the first nonlinear equation in (1.14) vanish, and that first nonlinear equation yields (1.19). At the same time, the second nonlinear equation in (1.14) vanishes altogether. The nonlinear equation (1.19) with real-valued $Q(x, t)$ and $P(x, t) \equiv 0$ is known [22, 23] as the Sawada–Kotera equation. On the other hand, the nonlinear system (1.14) with real-valued $Q(x, t)$ and $P(x, t) \equiv$

$Q_x(x,t)$ is known [23, 26] as the Kaup–Kupershmidt equation. While the KdV equation (1.9) describes surface water waves in narrow, shallow canals, the Sawada–Kotera and Kaup–Kupershmidt equations in (1.19) describe the propagation of steeper surface water waves of shorter wavelength in narrow, shallow canals.

1.4 A special case of the third-order system

A special case of the third-order linear ordinary differential operator is given by

$$L = iD^3 - i[q(x)\,D + D\,q(x)] + p(x), \tag{1.20}$$

where $i := \sqrt{-1}$ and the coefficients $q(x)$ and $p(x)$ are complex-valued functions of the independent variable x. By using

$$L\phi = ik^3\phi, \tag{1.21}$$

the third-order linear differential operator in (1.20) yields the third-order ODE

$$\phi''' - 2q(x)\phi' - [q'(x) + ip(x)]\,\phi = k^3\,\phi, \qquad x \in \mathbb{R}, \tag{1.22}$$

where ϕ is the wavefunction, and $q(x)$ and $p(x)$ are the potentials. In case $q(x)$ and $p(x)$ are real valued, the linear operator L given in (1.20) becomes selfadjoint, and consequently, its spectral analysis is simpler compared to the analysis of the third-order nonselfadjoint linear operator L appearing in (1.12).

The time-dependent version of (1.22) is given by

$$\phi''' - 2q(x,t)\,\phi' - [q_x(x,t) + ip(x,t)]\,\phi = k^3\phi, \qquad x \in \mathbb{R}, \tag{1.23}$$

where $q(x,t)$ and $p(x,t)$ denote the time-dependent potentials. If the potentials $q(x,t)$ and $p(x,t)$ are real valued, then (1.23) is associated with the integrable system of NPDEs given by

$$\begin{cases} q_t = -3p_x, \\ p_t = -q_{xxx} + 8qq_x. \end{cases} \tag{1.24}$$

The nonlinear system (1.24) is obtained from the operator equation (1.16) by using the Lax pair (L, A), where we have

$$\begin{cases} L = iD^3 - i(q\,D + D\,q) + p, \\ A = i(3D^2 - 4q). \end{cases} \tag{1.25}$$

In other words, the use of (1.25) in the operator equation (1.16) yields the nonlinear system (1.24). The system (1.24) is known [32] as the bad Boussinesq system, and it models the propagation of long water waves of small amplitude [3]. It is possible to eliminate p in (1.24) after taking the x-derivative of the first equation in (1.24) and by taking the t-derivative of the second equation there. This yields the single nonlinear evolution equation

$$q_{tt} - 3q_{xxxx} + 12(q^2)_{xx} = 0, \tag{1.26}$$

which is known [32] as the bad Boussinesq equation. In (1.26), if the sign of the coefficient of q_{xxxx} is changed, we obtain the good Boussinesq equation [32] given by

$$q_{tt} + 3q_{xxxx} + 12(q^2)_{xx} = 0. \tag{1.27}$$

In general, the bad and good Boussinesq equations describe [1,2,11] the propagation of long waves on the surface of shallow water.

1.5 The Inverse Scattering Transform method

The NPDEs associated with a Lax pair (L, A) are known as integrable in the sense of the IST (Inverse Scattering Transform) method, as Figure 1.3 depicts. For example, for the KdV equation (1.9) we have the diagram given in Figure 1.3.

The interpretation of the diagram in Figure 1.3 is as follows. We would like to solve the IVP (initial-value problem) for the NPDE, which is the KdV equation (1.9). In other words, we would like to determine $V(x,t)$ satisfying the NPDE with $V(x,t)$ at $t = 0$ being equal to the initial profile $V(x,0)$. Toward this goal, we use the following three steps:

(1) We associate the initial profile $V(x,0)$ with the initial scattering data $S(k,0)$. This is done by analyzing the direct scattering problem for the second-order linear ODE

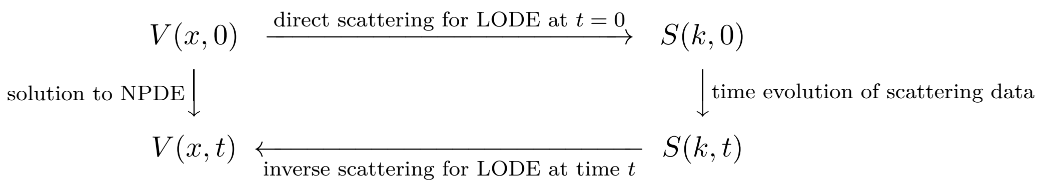

Figure 1.3: The inverse scattering transform for the KdV equation.

in (1.8). In Figure 1.3, this is indicated as solving the direct scattering problem for the LODE (linear ordinary differential equation) at $t = 0$. This involves finding some particular solutions to (1.8). It also involves the determination of the initial scattering data $S(k, 0)$ from the spacial asymptotics of those particular solutions.

(2) The second step involves finding the appropriate time evolution of the scattering data, i.e. the determination of the time-evolved scattering data $S(k, t)$ from $S(k, 0)$. This time evolution is specific to the associated NPDE. Normally, we expect the mathematical description of that time evolution to be rather simple.

(3) From the time-evolved scattering data $S(k, t)$, one recovers the time-evolved potential $V(x, t)$. This involves [20] solving the inverse scattering problem for the linear ODE in (1.8) at the fixed time t. It is an amazing fact that the time-evolved potential $V(x, t)$ is a solution to the IVP associated with (1.9). In other words, $V(x, t)$ is a solution to (1.9) and it reduces to the initial profile $V(x, 0)$ at $t = 0$.

This three-step process of solving an IVP for a NPDE is known as the IST method, and the corresponding NPDE is then called integrable in the sense of the IST.

1.6 A method for solving the inverse scattering problem for the third-order ODE

As already indicated, our main goal is to solve the inverse scattering problem for (1.1), and we accomplish this without requiring the corresponding linear operator in (1.12) to be selfadjoint.

The special third-order ODE (1.22) with the selfadjointness has been analyzed by Deift, Tomei, and Trubowitz in their important paper [15] under certain restrictions on the corresponding scattering data set. The restrictions used in [15] consist of the assumption that the two secondary reflection coefficients associated with (1.22) are identically equal to zero and that the two associated transmission coefficients are identically equal to one. The two secondary reflection coefficients and the two transmission coefficients are part of the scattering data set for (1.22). Those restrictions in [15] enabled Deift, Tomei, and Trubowitz to analyze the inverse scattering problem for the special third-order linear ODE (1.22) by using the techniques already known [13, 16, 19, 35] for the second-order linear ODE (1.6). Consequently, Deift, Tomei, and Trubowitz developed in [15] a solution technique to solve the IVP for the bad Boussinesq equation (1.26) under their assumed restrictions.

In solving the inverse scattering problem for (1.1), we have been inspired by the approach used in [15]. On the other hand, the method developed in this thesis is by no means a straightforward generalization of the method of [15]. In particular, our method is not restricted to the selfadjoint case studied in [15], and we do not impose the severe restrictions of setting the two transmission coefficients identically equal to one. Furthermore, we do not restrict ourselves to the necessity of an analytic continuation across a particular set of two half lines in the complex k-plane, as done in [15]. In our thesis work, by allowing a meromorphic continuation across those two half lines in the complex k-plane, we no longer need to require that the two transmission coefficients are identically equal to one.

For the history of the inverse scattering problem for the third-order operator and other approaches, we refer the reader to [8–10, 12, 23, 24] and the references therein. In those references, the formulation of the Riemann–Hilbert problem is not done on a single full line in the complex k-plane, but on the set of two full lines. It is more challenging to solve a Riemann–Hilbert problem formulated on multiple full lines than on a single full line. Furthermore, it is unclear how to establish a Fourier transformation on multiple full lines.

Consequently, in those other approaches it has been difficult to solve the corresponding inverse scattering problem and obtain some concrete results.

The bound states for (1.1) occur at certain k-values in the complex k-plane where the corresponding transmission coefficients have poles. The assumption that the transmission coefficients are identically equal to one, which is used in [15], results in the severe restriction that the special ODE considered in (1.22) does not have any bound states. That restriction also results in the assumption that the time-evolved special ODE (1.23) does not have any bound states. The assumption of no bound states for (1.23) is equivalent to the assumption that the corresponding bad Boussinesq equation (1.26) does not possess any soliton-type solutions. As a matter of fact, soliton-type solutions to (1.26) make up the important class of closed-form solutions that can be expressed explicitly in terms of elementary functions of x and t.

In our analysis of the inverse scattering problem for (1.1), we do not assume that the two transmission coefficients are identically equal to one, and hence our method applies in the presence of bound states. In particular, our work yields the IST method for each of the nonlinear systems (1.14), (1.19), (1.24), and (1.26). Consequently, our method is capable of producing soliton-type solutions to (1.14), (1.19), (1.24), and (1.26).

1.7 The organization of the thesis

This thesis is organized as follows. In Chapter 2, we provide a physical description of the scattering phenomena associated with (1.1). We introduce two particular solutions to (1.1), which we refer to as the Jost solutions, and we establish their relevant properties such as their k-domains, their continuity and analyticity in k, and their spacial asymptotics. Through those spacial asymptotics we introduce the scattering coefficients for (1.1). We then present the adjoint equation related to (1.1), and with the help of the Jost solutions to the adjoint equation we construct two additional solutions to (1.1). We also establish the relevant properties of those two solutions. We refer to the two Jost solutions to (1.1) and those two additional solutions as the four fundamental solutions. We then establish the relevant properties of the four fundamental solutions. In Chapter 3, using the spacial asymptotics of the two Jost solutions to the adjoint equation, we introduce the adjoint scattering coefficients. We establish the relationships between the scattering coefficients for (1.1) and the adjoint scattering coefficients. We then establish the small and large k-asymptotics of the four fundamental solutions to (1.1). In a similar manner, we determine the small and large k-asymptotics for the scattering coefficients for (1.1). Finally in Chapter 3, we describe the bound-state solutions to (1.1) and introduce the dependency constant for each bound state for (1.1). In Chapter 4, we describe the inverse scattering problem for (1.1). We establish the Riemann–Hilbert problem related to the inverse scattering problem for (1.1). We describe how the two potentials in (1.1) are recovered from the solution to the relevant Riemann–Hilbert problem. Finally, we mention how a Fourier transformation on the Riemann–Hilbert problem leads to a system of Marchenko integral equations associated with (1.1). The analysis of the Marchenko system and the recovery of the two potentials in (1.1) from the solution to the Marchenko system will be published elsewhere.

CHAPTER 2

THE DIRECT SCATTERING PROBLEM: PART I

In Chapter 1, we have introduced the third-order equation (1.1) along with the descriptions of the direct and inverse scattering problems associated with (1.1). In this chapter, we introduce certain particular solutions to (1.1), from which we construct a fundamental set of solutions to (1.1) at each k-value in the complex plane $\mathbb{C}$. We recall that a fundamental set of solutions for (1.1) contains three linearly independent solutions. We can obtain the general solution to (1.1) by using a linear combination of the particular solutions in the fundamental set. Recall that, for simplicity, we assume that the potentials $Q(x)$ and $P(x)$ appearing in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$, although our results are valid under weaker conditions on the potentials.

In the current chapter, we explain how the scattering phenomena associated with (1.1) occurs. We provide a physical description of the scattering for (1.1) by introducing the physical solutions $F(k,x)$ and $G(k,x)$, where those physical solutions to (1.1) are closely related to the particular solutions $f(k,x)$ and $g(k,x)$, respectively, which we first construct.

Since (1.1) is not associated with a selfadjoint linear differential operator, in order to construct a fundamental set of solutions to (1.1), we make use of certain particular solutions to the adjoint equation corresponding to (1.1). Hence, in our chapter we introduce the adjoint equation related to (1.1) and construct particular solutions to that adjoint equation. Then, by using the 2-Wronskians of those particular solutions to the adjoint equation, we are able to construct two additional particular solutions to (1.1) besides the particular solutions $f(k,x)$ and $g(k,x)$. We denote those two additional particular solutions to (1.1) as $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$. We also introduce the 3-Wronskian of three functions, which helps us determine whether any three solutions for (1.1) are linearly independent or linearly dependent. In Chapter 3, we will use the 3-wronskians of various solutions to (1.1) to determine the basic relevant relationships among the scattering coefficients for (1.1) and the adjoint scattering coefficients.

2.1 The direct scattering problem for (1.1)

The main goal of this thesis is to develop an effective mathematical method to solve the inverse scattering problem for (1.1). For this we need to understand the direct scattering problem for (1.1) well. In particular, we need to identify the scattering data set to be used as input to solve the corresponding inverse scattering problem for (1.1).

The scattering data set for (1.1) consists of the scattering coefficients and the bound-state information. In this chapter, our emphasis is the description of the relevant scattering phenomena and the construction of the scattering coefficients.

The scattering coefficients for (1.1) are obtained by using the spacial asymptotics of two of the particular solutions to (1.1). By exploiting the analogy with the second-order case, i.e. with the Schrödinger equation (1.6), we refer to those two particular solutions as the Jost solutions, and we denote them by $f(k,x)$ and $g(k,x)$, respectively. We refer to $f(k,x)$ as the Jost solution from the left and $g(k,x)$ as the Jost solution from the right. The terminology related to the left and right will become clear when we describe the scattering phenomena associated with (1.1). For simplicity, we use the term the left Jost solution to mean the Jost solution from the left, and we say the right Jost solution as an equivalent expression for the Jost solution from the right. We show that the k-domain of $f(k,x)$ is the region $\overline{\Omega_1}$, which corresponds to the closure of the sector Ω_1 in the complex k-plane, as shown in Figure 2.1. Similarly, we show that the k-domain of $g(k,x)$ is the region $\overline{\Omega_3}$, which corresponds to the closure of the sector Ω_3 in the complex k-plane, as shown in Figure 2.1.

It is convenient to describe the sectors Ω_1 and Ω_3 with the help of the special complex constant z

$$z := e^{2\pi i/3}, \tag{2.1}$$

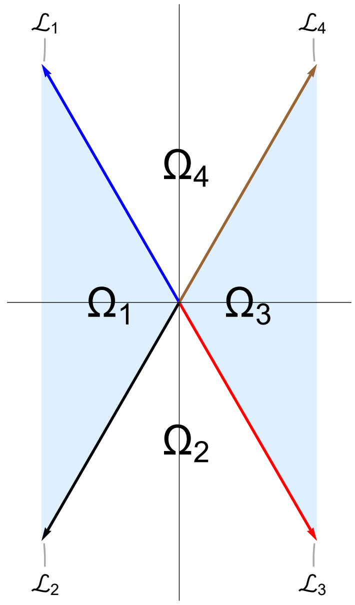

Figure 2.1: The k-domains $\overline{\Omega}_1$ and $\overline{\Omega}_3$ of $f(k,x)$ and $g(k,x)$, respectively.

which corresponds to a cube root of unity. Note that z satisfies

$$z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \tag{2.2}$$

$$z^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \tag{2.3}$$

$$z^3 = 1, \tag{2.4}$$

$$1 + z + z^2 = 0. \tag{2.5}$$

We partition the complex k-plane into four sectors denoted by $\Omega_1, \Omega_2, \Omega_3, \Omega_4$, respectively, as shown in Figure 2.1. Note that Ω_1 is the open region in the complex k-plane with the argument of k satisfying

$$\frac{2\pi}{3} < \arg[k] < \frac{4\pi}{3}. \tag{2.6}$$

We obtain $\overline{\Omega_1}$, the closure of Ω_1, by including the boundary of Ω_1. Hence, $\overline{\Omega}_1$ is described by

$$\frac{2\pi}{3} \leq \arg[k] \leq \frac{4\pi}{3}. \tag{2.7}$$

In the sector Ω_1, as shown in Figure 2.1, we use $\mathcal{L}_1$ to denote its directed upper boundary and use $\mathcal{L}_2$ to denote its directed lower boundary. The directions of $\mathcal{L}_1$ and $\mathcal{L}_2$ are indicated by an arrow in Figure 2.1. Note that Ω_3 is the sector in the complex k-plane with the argument of k satisfying

$$-\frac{\pi}{3} < \arg[k] < \frac{\pi}{3}. \tag{2.8}$$

Similarly, we obtain $\overline{\Omega_3}$ by including the boundary of the open set Ω_3. Hence, $\overline{\Omega_3}$ is described by

$$-\frac{\pi}{3} \leq \arg[k] \leq \frac{\pi}{3}. \tag{2.9}$$

In the sector Ω_3, as shown in Figure 2.1, we use $\mathcal{L}_4$ to denote its directed upper boundary and use $\mathcal{L}_3$ to denote its directed lower boundary. The directions of $\mathcal{L}_3$ and $\mathcal{L}_4$ are indicated in Figure 2.1. As mentioned already, the boundaries $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ are the directed half lines, and they can be parameterized in terms of the real parameter $s \in [0, +\infty)$ as

$$\mathcal{L}_1: \quad k = zs, \tag{2.10}$$

$$\mathcal{L}_2: \quad k = z^2 s, \tag{2.11}$$

$$\mathcal{L}_3: \quad k = -zs, \tag{2.12}$$

$$\mathcal{L}_4: \quad k = -z^2 s, \tag{2.13}$$

where we recall that z is the special complex number appearing in (2.1). The two remaining sectors Ω_2 and Ω_4 in Figure 2.1 are described in similar manners to Ω_1 and Ω_3. In particular, the sector Ω_2 has the parametrization

$$\frac{4\pi}{3} < \arg[k] < \frac{5\pi}{3}, \tag{2.14}$$

with the directed boundaries $\mathcal{L}_2$ and $\mathcal{L}_3$. The sector Ω_4 is parametrized by using

$$\frac{\pi}{3} < \arg[k] < \frac{2\pi}{3}, \tag{2.15}$$

with the directed boundaries $\mathcal{L}_1$ and $\mathcal{L}_4$. The closure of the sectors Ω_2 and Ω_4 are similarly defined by including their boundaries.

2.2 The general solution to (1.1)

Since the potentials $Q(x)$ and $P(x)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$, as $x \to \pm\infty$ any solution to the third-order ODE (1.1) behaves asymptotically as a solution to the corresponding "unperturbed equation"

$$\mathring{\psi}''' = k^3 \mathring{\psi}, \qquad x \in \mathbb{R}, \tag{2.16}$$

where $\mathring{\psi}(k,x)$ is the unperturbed wavefunction. The unperturbed equation in (2.16) is obtained from (1.1) by setting $Q(x) \equiv 0$ and $P(x) \equiv 0$. For each fixed $k \in \mathbb{C}$, the general solution to (2.16) is given by

$$\mathring{\psi}(k,x) = c_1(k)\, e^{kx} + c_2(k)\, e^{zkx} + c_3(k)\, e^{z^2kx}, \qquad x \in \mathbb{R}, \tag{2.17}$$

where the coefficients $c_1(k)$, $c_2(k)$, $c_3(k)$ are arbitrary constants that can only depend on k.

The general solution to (1.1) can be obtained by using the method of variation of parameters [14] on (1.1). For this we assume that the general solution to (1.1) is obtained from (2.17) by allowing $c_1(k)$, $c_2(k)$, $c_3(k)$ to depend on the independent variable x, i.e. by letting

$$\psi(k,x) = c_1(k,x)\, e^{kx} + c_2(k,x)\, e^{zkx} + c_3(k,x)\, e^{z^2kx}, \qquad x \in \mathbb{R}, \tag{2.18}$$

and by determining the restrictions on the coefficients $c_1(k,x)$, $c_2(k,x)$, $c_3(k,x)$ so that the right-hand side of (2.18) satisfies (1.1).

As already indicated, any solution to (1.1) has the asymptotic behavior similar to the right-hand side of (2.17). Consequently, the asymptotic behavior of any solution to (1.1) at each fixed $k \in \mathbb{C}$ is given by

$$\psi(k,x) = \begin{cases} a_1(k)\, e^{kx} + a_2(k)\, e^{zkx} + a_3(k)\, e^{z^2kx} + o(1), & x \to +\infty, \\ a_4(k)\, e^{kx} + a_5(k)\, e^{zkx} + a_6(k)\, e^{z^2kx} + o(1), & x \to -\infty, \end{cases} \tag{2.19}$$

for some appropriate coefficients $a_1(k)$, $a_2(k)$, $a_3(k)$, $a_4(k)$, $a_5(k), a_6(k)$.

2.3 The left and right Jost solutions to (1.1)

We define the particular solution $f(k,x)$ to (1.1), namely the left Jost solution to (1.1), by imposing three specific restrictions. Those three restrictions are obtained by choosing the coefficients $a_1(k)$, $a_2(k)$, $a_3(k)$ in (2.19) as

$$a_1(k) \equiv 1, \quad a_2(k) \equiv 0, \quad a_3(k) \equiv 0. \tag{2.20}$$

Since the potentials $Q(x)$ and $P(x)$ in (1.1) belong to $\mathcal{S}(\mathbb{R})$, it can directly be verified that $f(k,x)$ satisfies the spacial asymptotics

$$\begin{cases} f(k,x) = e^{kx}\left[1+o(1)\right], \qquad x \to +\infty, \\ \\ f'(k,x) = k\,e^{kx}\left[1+o(1)\right], \qquad x \to +\infty, \\ \\ f''(k,x) = k^2\,e^{kx}\left[1+o(1)\right], \qquad x \to +\infty, \end{cases} \tag{2.21}$$

where we recall that we use the prime to denote the x-derivative. The remaining coefficients $a_4(k), a_5(k), a_6(k)$ in (2.19) are then determined by the potentials $Q(x)$ and $P(x)$, and those three coefficients are related to the scattering coefficients for (1.1).

It is convenient to express the left Jost solution $f(k,x)$ as

$$f(k,x) = e^{kx} u(k,x), \tag{2.22}$$

with the help of the auxiliary function $u(k,x)$. Using (2.22) in (2.21) we see that $u(k,x)$ satisfies the third-order ODE given by

$$u''' + 3ku'' + [3k^2 + Q(x)]u' + [kQ(x) + P(x)]u = 0, \qquad x \in \mathbb{R}. \tag{2.23}$$

By using (2.22) in (2.21), we see that $u(k,x)$ satisfies the spacial asymptotics

$$\begin{cases} u(k,x) = 1 + o(1), \qquad x \to +\infty, \\ \\ u'(k,x) = o(1), \qquad x \to +\infty, \\ \\ u''(k,x) = o(1), \qquad x \to +\infty. \end{cases} \tag{2.24}$$

In the next theorem, we present the integral equation for $u(k,x)$, which is useful to determine all the relevant properties of $u(k,x)$. After we obtain those properties for $u(k,x)$, we also have all the relevant properties of the left Jost solution $f(k,x)$ by using (2.22).

Theorem 2.3.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Then, the quantity $u(k,x)$ appearing (2.22) satisfies the integral relation*

$$u(k,x) = 1 - \int_x^\infty dy\, D(k,x-y)\,[Q'(y) - P(y)]\, u(k,y) + \int_x^\infty dy\, E(k,x-y)\, Q(y)\, u(k,y), \tag{2.25}$$

where the quantities $D(k,y)$ and $E(k,y)$ are defined as

$$D(k,y) := \frac{1}{3k^2}\left[1 + z e^{(z-1)ky} + z^2 e^{(z^2-1)ky}\right], \tag{2.26}$$

$$E(k,y) := \frac{1}{3k}\left[1 + z^2 e^{(z-1)ky} + z e^{(z^2-1)ky}\right], \tag{2.27}$$

with z being the special constant appearing in (2.1).

Proof. By using the method of variation of parameters [14] on the third-order ODE (2.23), we convert (2.23) and (2.24) into the integral equation given in (2.25). ■

After introducing the left Jost solution $f(k,x)$ to (1.1), we now describe the construction of the solution $g(k,x)$. By choosing the coefficients $a_4(k), a_5(k), a_6(k)$ in (2.19) as

$$a_4(k) \equiv 1, \quad a_5(k) \equiv 0, \quad a_6(k) \equiv 0, \tag{2.28}$$

we obtain the particular solution to (1.1), which is the right Jost solution $g(k,x)$. Since the potentials $Q(x)$ and $P(x)$ in (1.1) belong to $\mathcal{S}(\mathbb{R})$, it can directly be verified that $g(k,x)$ satisfies the spacial asymptotics

$$\begin{cases} g(k,x) = e^{kx}\,[1 + o(1)]\,, & x \to -\infty, \\ g'(k,x) = k\, e^{kx}\,[1 + o(1)]\,, & x \to -\infty, \\ g''(k,x) = k^2\, e^{kx}\,[1 + o(1)]\,, & x \to -\infty. \end{cases} \tag{2.29}$$

After the choice in (2.28), the remaining coefficients $a_1(k)$, $a_2(k), a_3(k)$ in (2.19) are then determined by the potentials $Q(x)$ and $P(x)$, and those three coefficients are related to the scattering coefficients for (1.1).

As an analogy to (2.22), it is convenient to express the right Jost solution $g(k,x)$ as

$$g(k,x) = e^{kx} v(k,x), \tag{2.30}$$

by using the auxiliary function $v(k,x)$. By using (2.30) in (1.1), we see that $v(k,x)$ satisfies the same third-order ODE (2.23) satisfied by $u(k,x)$. In other words, we have

$$v''' + 3kv'' + [3k^2 + Q(x)]v' + [kQ(x) + P(x)]v = 0, \qquad x \in \mathbb{R}. \tag{2.31}$$

By using (2.30) in (2.29), we see that $v(k,x)$ satisfies the spacial asymptotics

$$\begin{cases} v(k,x) = 1 + o(1), \qquad x \to -\infty, \\ v'(k,x) = o(1), \qquad x \to -\infty, \\ v''(k,x) = o(1), \qquad x \to -\infty. \end{cases} \tag{2.32}$$

In the next theorem, we present the integral equation for $v(k,x)$, which is useful to determine all the relevant properties of $v(k,x)$ and in turn all the relevant properties of $g(k,x)$.

Theorem 2.3.2. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Then, the quantity $v(k,x)$ appearing in (2.30) satisfies the integral relation*

$$v(k,x) = 1 + \int_{-\infty}^{x} dy\, D(k, x-y)\, [Q'(y) - P(y)]\, v(k,y) - \int_{-\infty}^{x} dy\, E(k, x-y)\, Q(y)\, v(k,y), \tag{2.33}$$

where $D(x,y)$ and $E(x,y)$ are the quantities in (2.26) and (2.27), respectively.

Proof. The proof is similar to the proof used in Theorem 2.3.1. By using the method of variation of parameters [14], the third-order ODE (2.31) and the asymptotic conditions in (2.32) are converted into the integral equation given in (2.33). ∎

2.4 The k-domains of the left and right Jost solutions

The method of converting a differential equation supplemented by a set of asymptotic conditions into an integral equation is a well established technique [14, 16, 19]. Once the conversion to the integral equation is accomplished, the solution is obtained by solving the corresponding integral equation iteratively. This iteration involves the representation of the solution to the integral equation as a uniformly convergent infinite series. The iterative technique also allows us to establish the existence and uniqueness of any particular solution to a differential equation supplemented by some asymptotic conditions. In particular, it helps us determine various properties of that particular solution such as the continuity in k or x, the analyticity in k, and the small and large asymptotics in k. The continuity and analyticity properties of the solution are established with the help of the following theorem [13, 14, 16], which we state without a proof.

Theorem 2.4.1. *Let $u(k,x)$ be represented as an infinite series of the form*

$$u(k,x) = \sum_{j=0}^{\infty} u_j(k,x), \qquad x \in \mathbb{R}, \tag{2.34}$$

where $u_j(k,x)$ is the jth term of the series. We have the following:

(a) *Assume that each $u_j(k,x)$ is continuous in $x \in \mathbb{R}$ for some fixed k-value in $\mathbb{C}$ and that the series in (2.34) is uniformly convergent for $x \in \mathbb{R}$ at that k-value. Then, $u(k,x)$ itself is continuous in $x \in \mathbb{R}$ at that k-value.*

(b) *Assume that, for each fixed $x \in \mathbb{R}$, each of the terms $u_j(k,x)$ is analytic in every compact subset in an open connected set Ω in the complex k-plane $\mathbb{C}$. Further, assume that the series in (2.34) is uniformly convergent when $k \in \Omega$ at each fixed $x \in \mathbb{R}$. Then, $u(k,x)$ itself is analytic in $k \in \Omega$ for each fixed $x \in \mathbb{R}$.*

(c) *Assume that, for each fixed $x \in \mathbb{R}$, each of the terms $u_j(k,x)$ is continuous in the closure $\overline{\Omega}$ of the set Ω in the complex k-plane $\mathbb{C}$. Further, assume that the series in (2.34) is uniformly convergent when $k \in \overline{\Omega}$ at each fixed $x \in \mathbb{R}$. Then, $u(k,x)$ is continuous in $k \in \overline{\Omega}$ for each fixed $x \in \mathbb{R}$.*

For each fixed $x \in \mathbb{R}$, we obtain the k-domain of the auxiliary function $u(k,x)$ appearing in (2.22) by solving the integral equation (2.25) iteratively. This involves the identification of the k-values in the complex plane $\mathbb{C}$ for which the solution to (2.25) exists. This is done by assuring that the exponential terms in the integrand do not blow up as $x \to +\infty$. Solving the integral equation (2.25) iteratively can be accomplished by representing its solution $u(k,x)$ to (2.25) as a uniformly convergent infinite series as in (2.34) and by identifying the k-values in the complex plane $\mathbb{C}$ at which the uniform convergence holds.

The advantage of using the auxiliary function $u(k,x)$ in (2.25) rather than the left Jost solution $f(k,x)$ is that $u(k,x)$ has the large k-asymptotics equal to 1, while $f(k,x)$ has the large k-asymptotics e^{kx}. As a result, the representation of $u(k,x)$ as a uniformly convergent series as in (2.34) is easier to handle than representing $f(k,x)$ as a uniformly convergent infinite series.

In the next theorem, we present certain properties of the solution $u(k,x)$ to (2.23) satisfying the asymptotics (2.24).

Theorem 2.4.2. *Assume that the potentials $Q(x)$ and $P(x)$ in* (1.1) *belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Then, we have the following:*

(a) *The integral equation* (2.25) *has a solution and that solution is unique.*

(b) *Consequently, the solution $u(k,x)$ to* (2.23) *satisfying the asymptotics* (2.24) *exists and is unique.*

(c) *For each fixed $x \in \mathbb{R}$, the solution $u(k,x)$ is analytic in $k \in \Omega_1$ and continuous in $\overline{\Omega}_1$, where we recall that Ω_1 is the open sector shown in Figure 2.1 and $\overline{\Omega}_1$ is the closure of Ω_1.*

(d) *For each fixed $k \in \overline{\Omega}_1$, the solution $u(k,x)$ is continuous in $x \in \mathbb{R}$.*

Proof. The proof of (a) is obtained by solving (2.25) iteratively and by representing its solution as a uniformly convergent series as in (2.34). We remark that (b) is a direct consequence of (a). The proof of (c) is obtained by applying Theorem 2.4.1(b) and Theorem 2.4.1(c) on the infinite series representation of the solution to (2.25) as in (2.34). Finally, the proof of (d) is obtained by applying Theorem 2.4.1(c) on the uniformly convergent series representation of the solution to (2.25). ■

Using (2.22) and Theorem 2.4.2, we have the following corollary.

Corollary 2.4.2.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Then, we have the following:*

(a) *The left Jost solution $f(k,x)$ to (1.1) satisfying (2.21) exists and is unique.*

(b) *For each fixed $x \in \mathbb{R}$, the left Jost solution $f(k,x)$ is analytic in $k \in \Omega_1$ and continuous in $\overline{\Omega}_1$.*

(c) *For each fixed $k \in \overline{\Omega}_1$, the left Jost solution $f(k,x)$ is continuous in $x \in \mathbb{R}$.*

In a similar way, in the next theorem we present certain relevant properties of the solution $v(k,x)$ to (2.31) satisfying the asymptotics (2.32).

Theorem 2.4.3. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Then, we have the following:*

(a) *The integral equation (2.33) has a solution and that solution is unique.*

(b) *Consequently, the solution $v(k,x)$ to (2.31) satisfying the asymptotics (2.32) exists and is unique.*

(c) *For each fixed $x \in \mathbb{R}$, the solution $v(k,x)$ is analytic in $k \in \Omega_3$ and continuous in $\overline{\Omega}_3$, where we recall that Ω_3 is the open sector shown in Figure 2.1 and $\overline{\Omega}_3$ is the closure of Ω_3.*

(d) *For each fixed $k \in \overline{\Omega}_3$, the solution $v(k,x)$ is continuous in $x \in \mathbb{R}$.*

Proof. The proof is analogous to the proof of Theorem 2.4.2. This is accomplished by representing the solution to (2.33) as a uniformly convergent infinite series, by solving (2.33) via iteration, and by applying Theorem 2.4.1. ■

Using (2.30) and Theorem 2.4.3, we have the following corollary.

Corollary 2.4.3.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Then, we have the following:*

(a) *The right Jost solution $g(k,x)$ to (1.1) satisfying (2.29) exists and is unique.*

(b) *For each fixed $x \in \mathbb{R}$, the solution $g(k,x)$ is analytic in $k \in \Omega_3$ and continuous in $\overline{\Omega}_3$.*

(c) *For each fixed $k \in \overline{\Omega}_3$, the solution $g(k,x)$ is continuous in $x \in \mathbb{R}$.*

2.5 The scattering coefficients for (1.1)

As we have seen in Corollaries 2.4.2.1 and 2.4.3.1, the specification of three of the six coefficients in (2.19) helps us construct the two particular solutions $f(k,x)$ and $g(k,x)$. The remaining three coefficients in (2.19) allow us to define the corresponding scattering coefficients for (1.1). Thus, we are able to define the scattering coefficients for (1.1) by using the spacial asymptotics of the Jost solutions $f(k,x)$ and $g(k,x)$ to (1.1).

In the next theorem, we use the spacial asymptotics of the left Jost solution $f(k,x)$ to describe the left scattering coefficients $T_{\mathrm{l}}(k)$, $L(k)$, and $M(k)$ for (1.1).

Theorem 2.5.1. *Assume that the potentials $Q(x)$ and $P(x)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Then, the left Jost solution $f(k,x)$ to (1.1) appearing in (2.21) has the spacial asymptotics as $x \to -\infty$ given by*

$$f(k,x) = \begin{cases} e^{kx}\, T_{\mathrm{l}}(k)^{-1}[1+o(1)] + e^{zkx} L(k)\, T_{\mathrm{l}}(k)^{-1}[1+o(1)], \qquad k \in \mathcal{L}_1, \\[2ex] e^{kx}\, T_{\mathrm{l}}(k)^{-1}[1+o(1)], \qquad k \in \Omega_1, \\[2ex] e^{kx}\, T_{\mathrm{l}}(k)^{-1}[1+o(1)] + e^{z^2kx} M(k)\, T_{\mathrm{l}}(k)^{-1}[1+o(1)], \qquad k \in \mathcal{L}_2, \end{cases} \tag{2.35}$$

where we have defined

$$T_{\mathrm{l}}(k)^{-1} := 1 + \int_{-\infty}^{\infty} dy \left[-\frac{Q'(y)-P(y)}{3k^2} + \frac{Q(y)}{3k} \right] u(k,y), \tag{2.36}$$

$$L(k)\, T_{\mathrm{l}}(k)^{-1} := \int_{-\infty}^{\infty} dy\, e^{i\sqrt{3}z^2ky} \left[-\frac{z}{3k^2}\left(Q'(y)-P(y)\right) + \frac{z^2}{3k} Q(y) \right] u(k,y), \tag{2.37}$$

$$M(k)\, T_{\mathrm{l}}(k)^{-1} := \int_{-\infty}^{\infty} dy\, e^{-i\sqrt{3}zky} \left[-\frac{z^2}{3k^2}\left(Q'(y)-P(y)\right) + \frac{z}{3k} Q(y) \right] u(k,y). \tag{2.38}$$

The domain of $T_{\rm l}(k)^{-1}$ *in* (2.36) *is* $\overline{\Omega}_1 \setminus \{0\}$, *where we recall that* Ω_1 *is the open sector defined in* (2.6) *and* $\overline{\Omega}_1$ *is the closure of* Ω_1. *The domain of* $L(k)\, T_{\rm l}(k)^{-1}$ *in* (2.37) *is* $\mathcal{L}_1 \setminus \{0\}$ *and the domain* $M(k)\, T_{\rm l}(k)^{-1}$ *in* (2.38) *is* $\mathcal{L}_2 \setminus \{0\}$, *where we recall that* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are the directed upper and lower boundaries of* Ω_1, *respectively, as shown in Figure 2.1.*

Proof. By letting $x \to -\infty$ in (2.25) and using (2.22), we obtain the asymptotics of $f(k, x)$ given in (2.35). The k-domains of the quantities on the right-hand sides of (2.36)–(2.38) are obtained by ensuring that the three integrals there are convergent. ■

As we see from Theorem 2.5.1, the asymptotics of the left Jost solution $f(k, x)$ as $x \to -\infty$ uniquely provides the coefficients $T_{\rm l}(k)^{-1}, L(k)\, T_{\rm l}(k)^{-1}$, and $M(k)\, T_{\rm l}(k)^{-1}$. From those three coefficients, we can uniquely determine $T_{\rm l}(k), L(k)$, and $M(k)$. We collectively refer to $T_{\rm l}(k), L(k)$, and $M(k)$ as the left scattering coefficients for (1.1). We refer to $T_{\rm l}(k)$ as the transmission coefficient from the left (or the left transmission coefficient, for short), refer to $L(k)$ as the primary reflection coefficient from the left (or the left primary reflection coefficient, for short), and refer to $M(k)$ as the secondary reflection coefficient from the left (or the left secondary reflection coefficient, for short).

In Theorem 2.5.1, with the help of the spacial asymptotics of the left Jost solution $f(k, x)$, we have introduced the three left scattering coefficients for (1.1). In the next theorem, in an analogous manner, we use the spacial asymptotics of the right Jost solution $g(k, x)$ to introduce the right scattering coefficients $T_{\rm r}(k), R(k)$, and $N(k)$ for (1.1).

Theorem 2.5.2. *Assume that the potentials* $Q(x)$ *and* $P(x)$ *belong to the Schwartz class* $\mathcal{S}(\mathbb{R})$. *Then, the right Jost solution* $g(k, x)$ *to* (1.1), *appearing in* (2.29) *has the spacial asymptotics as* $x \to +\infty$ *given by*

$$g(k,x) = \begin{cases} e^{kx}\, T_{\rm r}(k)^{-1}[1+o(1)] + e^{zkx} R(k)\, T_{\rm r}(k)^{-1}[1+o(1)], \qquad k \in \mathcal{L}_3, \\[2ex] e^{kx}\, T_{\rm r}(k)^{-1}[1+o(1)], \qquad k \in \Omega_3, \\[2ex] e^{kx}\, T_{\rm r}(k)^{-1}[1+o(1)] + e^{z^2kx} N(k)\, T_{\rm r}(k)^{-1}[1+o(1)], \qquad k \in \mathcal{L}_4, \end{cases} \tag{2.39}$$

where we have defined

$$T_{\rm r}(k)^{-1} := 1 + \int_{-\infty}^{\infty} dy \left[\frac{Q'(y) - P(y)}{3k^2} - \frac{Q(y)}{3k} \right] v(k,y), \tag{2.40}$$

$$R(k)\, T_{\rm r}(k)^{-1} := \int_{-\infty}^{\infty} dy\, e^{i\sqrt{3}z^2 ky} \left[\frac{z}{3k^2} \left(Q'(y) - P(y)\right) - \frac{z^2}{3k} Q(y) \right] v(k,y), \tag{2.41}$$

$$N(k)\, T_{\rm r}(k)^{-1} := \int_{-\infty}^{\infty} dy\, e^{-i\sqrt{3}zky} \left[\frac{z^2}{3k^2} \left(Q'(y) - P(y)\right) - \frac{z}{3k} Q(y) \right] v(k,y). \tag{2.42}$$

The domain of $T_{\rm r}(k)^{-1}$ *in* (2.40) *is* $\overline{\Omega}_3 \setminus \{0\}$, *where we recall that* Ω_3 *is the open sector defined in* (2.6) *and* $\overline{\Omega}_3$ *is the closure of* Ω_3. *The domain of* $R(k)\, T_{\rm r}(k)^{-1}$ *in* (2.41) *is* $\mathcal{L}_3 \setminus \{0\}$ *and the domain* $N(k)\, T_{\rm r}(k)^{-1}$ *in* (2.42) *is* $\mathcal{L}_4 \setminus \{0\}$, *where we recall that* $\mathcal{L}_3$ *and* $\mathcal{L}_4$ *are the directed lower and upper boundaries of* Ω_3, *respectively, as shown in Figure 2.1.*

Proof. The proof is similar to the proof of Theorem 2.5.1. By letting $x \to +\infty$ in (2.33) and using (2.30), we obtain the asymptotics of $g(k,x)$ given in (2.39). The restrictions on the k-domains of the quantities in (2.40)–(2.42) ensure that each of the integrals on the right-hand sides of (2.40)–(2.42) is convergent. ■

As demonstrated in Theorem 2.5.2, the asymptotics of the right Jost solution $g(k,x)$ as $x \to +\infty$ uniquely provides the coefficients $T_{\rm r}(k)^{-1}, R(k)\, T_{\rm r}(k)^{-1}$, and $N(k)\, T_{\rm r}(k)^{-1}$. From those three coefficients, we can uniquely determine $T_{\rm r}(k), R(k)$, and $N(k)$. We collectively refer to $T_{\rm r}(k), R(k)$, and $N(k)$ as the right scattering coefficients for (1.1). In particular, we refer to $T_{\rm r}(k)$ as the transmission coefficient from the right (or the right transmission coefficient, for short), refer to $R(k)$ as the primary reflection coefficient from the right (or the right primary reflection coefficient, for short), and refer to $N(k)$ as the secondary reflection coefficient from the right (or the right secondary reflection coefficient, for short).

From Theorems 2.5.1 and 2.5.2, we know that each of the six quantities $T_{\rm l}(k)^{-1}$, $L(k)\, T_{\rm l}(k)^{-1}$, $M(k)\, T_{\rm l}(k)^{-1}$, $T_{\rm r}(k)^{-1}$, $R(k)\, T_{\rm r}(k)^{-1}$, $N(k)\, T_{\rm r}(k)^{-1}$ in general has a singularity at $k = 0$. However, as we will see in Theorem 3.5.1 and Corollary 3.5.1.1, each of the six scattering coefficients $T_{\rm l}(k)$, $L(k)$, $M(k)$, $T_{\rm r}(k)$, $R(k)$, $N(k)$ is continuous at $k = 0$ and hence does not have a singularity there. Thus, with the help of Theorem 2.5.1 we see that the left primary reflection coefficient $L(k)$ is defined when $k \in \mathcal{L}_1$, the left secondary reflection coefficient $M(k)$ is defined when $k \in \mathcal{L}_2$, and the left transmission coefficient $T_{\rm l}(k)$ is defined when k is in the closure set $\overline{\Omega}_1$. Alternatively, we can assume that the left transmission coefficient $T_{\rm l}(k)$ is originally defined on the boundaries $\mathcal{L}_1$ and $\mathcal{L}_2$ and then meromorphically extended in k to the interior region Ω_1. Similarly, with the help of Theorem 2.5.2 we observe that the right primary reflection coefficient $R(k)$ is defined when $k \in \mathcal{L}_3$, the right secondary reflection coefficient $N(k)$ is defined when $k \in \mathcal{L}_4$, and the right transmission coefficient $T_{\rm r}(k)$ is defined when k is in the closure set $\overline{\Omega}_3$. Alternatively, we can assume that the right transmission coefficient $T_{\rm r}(k)$ is originally defined on the boundaries $\mathcal{L}_3$ and $\mathcal{L}_4$ and then meromorphically extended in k to the interior region Ω_3. For easy referencing, we summarize the k-domains of the six scattering coefficients for (1.1) as

$$\begin{cases} T_{\rm l}(k),\ k \in \overline{\Omega}_1; \quad L(k),\ k \in \mathcal{L}_1; \quad M(k),\ k \in \mathcal{L}_2, \\ T_{\rm r}(k),\ k \in \overline{\Omega}_3; \quad R(k),\ k \in \mathcal{L}_3; \quad N(k),\ k \in \mathcal{L}_4. \end{cases} \tag{2.43}$$

2.6 The scattering phenomena for (1.1)

Having described the six scattering coefficients $T_{\rm l}(k), L(k), M(k), T_{\rm r}(k), R(k), N(k)$ for (1.1), in this section we present their physical relevance. This is accomplished by presenting how the scattering phenomena for (1.1) occurs. This also clarifies why we use the terminology left and right in the description of the Jost solutions $f(k, x)$ and $g(k, x)$ and the six scattering coefficients $T_{\rm l}(k), L(k), M(k), T_{\rm r}(k), R(k), N(k)$.

In terms of the Jost solutions $f(k, x)$ and $g(k, x)$, we define the physical solutions $F(k, x)$ and $G(k, x)$, respectively, as

$$\begin{cases} F(k,x) := T_{\rm l}(k)\, f(k,x), \\ G(k,x) := T_{\rm r}(k)\, g(k,x). \end{cases} \tag{2.44}$$

We refer to $F(k, x)$ as the physical solution from the left (or the left physical solution, for short) and refer to $G(k, x)$ as the physical solution from the right (or the right physical

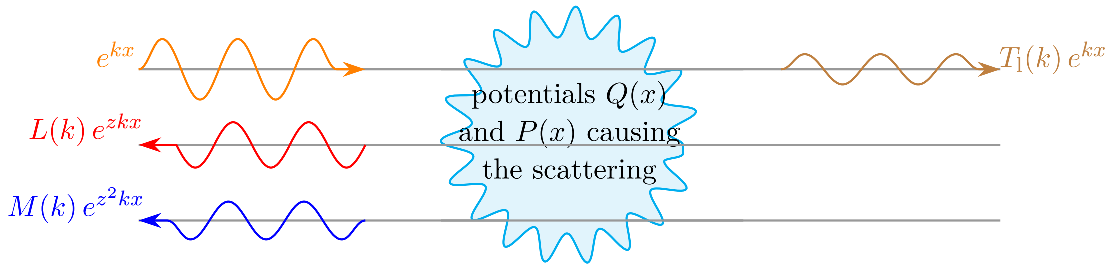

Figure 2.2: The scattering phenomena for (1.1) related to the physical solution $F(k, x)$.

solution, for short). Since (1.1) is a linear homogeneous ODE, due to the absence of a nonhomogeneous term, any constant multiple of a solution to (1.1) is also a solution. Since $T_{\mathrm{l}}(k)$ and $T_{\mathrm{r}}(k)$ do not contain the independent variable x and $f(k, x)$ and $g(k, x)$ are solutions to (1.1), from (2.44) it follows that $F(k, x)$ and $G(k, x)$ are also solutions to (1.1).

The following scenario, in analogy with the scattering scenario for the Schrödinger equation (1.6), explains why we refer to $F(k, x)$ and $G(k, x)$ as the physical solutions and also why we refer to $T_{\mathrm{l}}(k), L(k), M(k), T_{\mathrm{r}}(k), R(k), N(k)$ as the scattering coefficients. From (2.21), (2.35), and the first line of (2.44), we see that the physical solution $F(k, x)$ satisfies the spacial asymptotics as $x \to +\infty$

$$F(k, x) = T_{\mathrm{l}}(k)\, e^{kx}[1 + o(1)], \qquad k \in \overline{\Omega}_1, \tag{2.45}$$

and the spacial asymptotics as $x \to -\infty$

$$F(k, x) = \begin{cases} e^{kx}[1 + o(1)] + L(k)\, e^{zkx}[1 + o(1)], & k \in \mathcal{L}_1, \\ e^{kx}[1 + o(1)], & k \in \Omega_1, \\ e^{kx}[1 + o(1)] + M(k)\, e^{z^2kx}[1 + o(1)], & k \in \mathcal{L}_2. \end{cases} \tag{2.46}$$

The scattering phenomena for (1.1) associated with the spacial asymptotics of $F(k, x)$ is illustrated in Figure 2.2.

We interpret Figure 2.2 by relating it to the spacial asymptotics of $F(k, x)$ given in (2.45) and (2.46) as follows. From $x = -\infty$, we send the plane wave e^{kx} of unit amplitude onto the nonhomogenity described by the potentials $Q(x)$ and $P(x)$. That plane wave

e^{kx} incoming from the left interacts with the potentials and a part of it is transmitted to $x = +\infty$ but by remaining in the same channel as the incoming wave e^{kx}. The transmitted wave at $x = +\infty$ is given by $T_l(k)\, e^{kx}$, and hence it has the amplitude $T_l(k)$. Furthermore, that transmitted wave travels in the same direction as the incoming wave e^{kx} travels, namely it moves from the left to the right. Since the incoming plane wave is traveling from the left, i.e. from $x = -\infty$, it is appropriate to call $T_l(k)$ the transmission coefficient from the left. We interpret a part of the asymptotic solution, i.e. the part $L(k)\, e^{zkx}$, as the part of the wave reflected from the potentials $Q(x)$ and $P(x)$, where the reflection takes place in the channel described by e^{zkx}. We interpret $L(k)\, e^{zkx}$ as the reflected wave at $x = -\infty$ due to the interaction of the incoming plane wave e^{kx} with the nonhomogenity created by the potentials $Q(x)$ and $P(x)$. The wave $L(k)\, e^{zkx}$ travels from the right to the left, but because that reflected wave is caused by the incoming plane wave from the left, we refer to the reflection coefficient $L(k)$ as the reflection coefficient from the left. In our description of the scattering scenario, we choose to refer to the channel e^{zkx} as the primary reflection channel. Hence, we refer to $L(k)$ as the primary reflection coefficient from the left. In a similar manner, we interpret the asymptotic term $M(k)\, e^{z^2kx}$ as the wave reflected into the secondary channel resulting from the incoming plane wave e^{kx} of unit amplitude. We refer to the channel of e^{z^2kx} as the secondary reflection channel, and hence we call $M(k)\, e^{z^2kx}$ as the wave reflected into the secondary channel caused by the incoming wave e^{kx} from the left. Thus, it is appropriate to call $M(k)$ as the secondary reflection coefficient from the left. Consequently, we refer to the three scattering coefficients $T_l(k), L(k), M(k)$ as the scattering coefficients from the left or as the left scattering coefficients, for short. Similarly, it is appropriate to refer to the solution $F(k, x)$ to (1.1) as the physical solution from the left because it has an appropriate physical interpretation describing the scattering phenomena initiated by the unit amplitude incoming plane wave from the left and interacting with the potentials.

Having described the scattering phenomena for (1.1) associated with the spacial asymptotics of $F(k,x)$, we now present the scattering phenomena associated with the spacial asymptotics of $G(k,x)$. From (2.29), (2.39), and the second line of (2.44), we see that the physical solution $G(k,x)$ satisfies the spacial asymptotics as $x \to -\infty$

$$G(k,x) = T_{\rm r}(k)\, e^{kx}[1+o(1)], \qquad k \in \overline{\Omega}_3, \tag{2.47}$$

and the spacial asymptotics as $x \to +\infty$

$$G(k,x) = \begin{cases} e^{kx}[1+o(1)] + R(k)\, e^{zkx}[1+o(1)], & k \in \mathcal{L}_3, \\ e^{kx}[1+o(1)], \qquad k \in \Omega_3, \\ e^{kx}[1+o(1)] + N(k)\, e^{z^2kx}[1+o(1)], & k \in \mathcal{L}_4. \end{cases} \tag{2.48}$$

The scattering phenomena for (1.1) associated with the spacial asymptotics of $G(k,x)$ is illustrated in Figure 2.3.

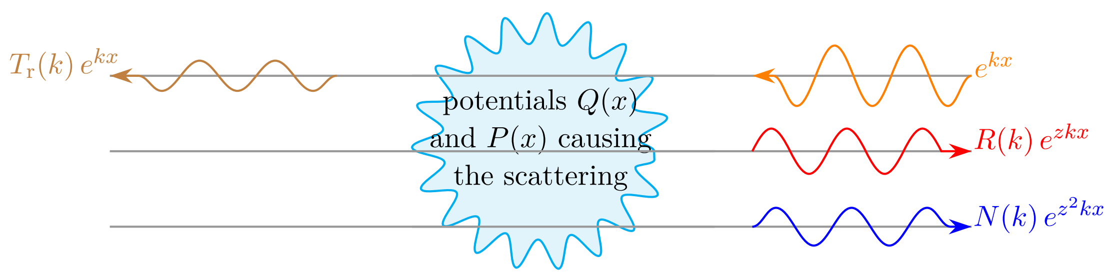

Figure 2.3: The scattering phenomena for (1.1) related to the physical solution $G(k,x)$.

We interpret Figure 2.3 by relating it to the spacial asymptotics of $G(k,x)$ given in (2.47) and (2.48) as follows. From $x = +\infty$, we send the plane wave e^{kx} of unit amplitude onto the nonhomogenity described by the potentials $Q(x)$ and $P(x)$. That incoming plane wave e^{kx} interacts with the potentials and a part of it is transmitted to $x = -\infty$ but by remaining in the same channel as the incoming wave e^{kx}. The transmitted wave at $x = -\infty$ is given by $T_{\rm r}(k)\, e^{kx}$, and hence it has the amplitude $T_{\rm r}(k)$. Furthermore, it travels in the same direction as the incoming wave travels, namely it moves from the right to the left. Since the incoming plane wave is coming from the right, i.e. from $x = +\infty$, it is appropriate to call $T_{\mathrm{r}}(k)$ the transmission coefficient from the right. We interpret a part of the asymptotic solution, i.e. the part $R(k)\, e^{zkx}$, as the part of the wave reflected from the potentials $Q(x)$ and $P(x)$, where the reflection takes place in the channel described by e^{zkx}. We interpret $R(k)\, e^{zkx}$ as the reflected wave at $x = +\infty$ due to the interaction of the incoming plane wave e^{kx} with the nonhomogenity created by the potentials $Q(x)$ and $P(x)$. The wave $R(k)\, e^{zkx}$ travels from the left to the right, but because that reflected wave originates from the incoming plane wave from the right, we refer to the reflection coefficient $R(k)$ as the reflection coefficient from the right. Since we refer to the channel e^{zkx} as the primary reflection channel, consequently we refer to $R(k)$ as the primary reflection coefficient from the right. In a similar manner, we interpret the asymptotic term $N(k)\, e^{z^2kx}$ as the wave reflected into the secondary reflection channel resulting from the incoming plane wave e^{kx} of unit amplitude. Since we refer to the e^{z^2kx} channel as the secondary reflection channel, consequently we call $N(k)\, e^{z^2kx}$ the wave reflected into the secondary reflection channel when the reflection is caused by the incoming wave e^{kx} from the right. It is thus appropriate to call $N(k)$ the secondary reflection coefficient from the right. Consequently, we refer to the three scattering coefficients $T_{\mathrm{r}}(k), R(k), N(k)$ as the scattering coefficients from the right or as the right scattering coefficients, for short. Similarly, it is appropriate to refer to the solution $G(k,x)$ to (1.1) as the physical solution from the right. This is because it has an appropriate physical interpretation and that physical interpretation describes the scattering phenomena initiated by the unit amplitude incoming plane wave from the right and interacting with the potentials.

Since we call $F(k,x)$ the physical solution from the left and call $G(k,x)$ the physical solution from the right, from (2.44) we see that it is appropriate to refer to $f(k,x)$ as the Jost solution from the left and refer to $g(k,x)$ as the Jost solution from the right.

2.7 The adjoint equation for (1.1)

The general solution to (1.1) at each fixed k-value in the complex plane $\mathbb{C}$ can be constructed by using a combination of three linearly independent solutions to (1.1). Toward this goal, we construct three linearly independent particular solutions to (1.1) at each k-value in $\mathbb{C}$. We already have the two particular solutions, namely the left Jost solution $f(k,x)$ with the k-domain $\overline{\Omega}_1$ and the right Jost solution $g(k,x)$ with the k-domain $\overline{\Omega}_3$. As a first step toward our goal, we construct a particular solution to (1.1) with the k-domain $\overline{\Omega}_2$ and we use $h^{\text{down}}(k,x)$ to denote that solution. Similarly, we construct another particular solution to (1.1) with the k-domain $\overline{\Omega}_4$ and we use $h^{\text{up}}(k,x)$ to denote that solution. The linear differential operator in (1.12) associated with (1.1) is not selfadjoint. Hence, we construct $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ with the help of the Jost solutions to the adjoint equation associated with (1.1).

We define the adjoint equation associated with (1.1) as

$$\overline{\psi}''' + \overline{Q}(x)\,\overline{\psi}' + \overline{P}(x)\,\overline{\psi} = k^3\,\overline{\psi}, \qquad x \in \mathbb{R}, \tag{2.49}$$

where $\overline{Q}(x)$ and $\overline{P}(x)$ are the adjoint potentials related to the potentials $Q(x)$ and $P(x)$ appearing in (1.1) as

$$\overline{Q}(x) := Q(x)^*, \quad \overline{P}(x) := Q'(x)^* - P(x)^*, \qquad x \in \mathbb{R}, \tag{2.50}$$

with the asterisk denoting complex conjugation. We remark that we use an overbar to identify the quantities associated with the adjoint equation, and we emphasize that the overbar does not denote complex conjugation.

The adjoint equation (2.49) has the Jost solutions $\overline{f}(k,x)$ and $\overline{g}(k,x)$ satisfying the analogs of (2.21) and (2.29), respectively. The left Jost solution $\overline{f}(k,x)$ to (2.49) is defined as the particular solution satisfying the spacial asymptotics

$$\begin{cases} \overline{f}(k,x) = e^{kx}\left[1+o(1)\right], \qquad x\to+\infty, \\ \\ \overline{f}'(k,x) = k\, e^{kx}\left[1+o(1)\right], \qquad x\to+\infty, \\ \\ \overline{f}''(k,x) = k^2\, e^{kx}\left[1+o(1)\right], \qquad x\to+\infty. \end{cases} \tag{2.51}$$

Similarly, the right Jost solution $\overline{g}(k,x)$ to (2.49) is defined as the particular solution satisfying the spacial asymptotics

$$\begin{cases} \overline{g}(k,x) = e^{kx}\left[1+o(1)\right], \qquad x\to-\infty, \\ \\ \overline{g}'(k,x) = k\, e^{kx}\left[1+o(1)\right], \qquad x\to-\infty, \\ \\ \overline{g}''(k,x) = k^2\, e^{kx}\left[1+o(1)\right], \qquad x\to-\infty. \end{cases} \tag{2.52}$$

As we see from (2.50), the adjoint potentials $\overline{Q}(x)$ and $\overline{P}(x)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ because we assume that the potentials $Q(x)$ and $P(x)$ belong to $\mathcal{S}(\mathbb{R})$. Furthermore, as seen from (2.21) and (2.51), the left Jost solutions $f(k,x)$ and $\overline{f}(k,x)$ have the same asymptotics as $x\to+\infty$. Hence, $f(k,x)$ and $\overline{f}(k,x)$ have similar properties. In particular, $f(k,x)$ and $\overline{f}(k,x)$ have the same k-domain $\overline{\Omega}_1$ in the complex plane $\mathbb{C}$. Similarly, as seen from (2.29) and (2.52), the right Jost solutions $g(k,x)$ and $\overline{g}(k,x)$ have the same asymptotics as $x\to-\infty$. Thus, $g(k,x)$ and $\overline{g}(k,x)$ have similar properties. In particular, they have the same k-domain $\overline{\Omega}_3$.

Let us remark that if $\psi(k,x)$ is a solution to (1.1), then $\psi(zk,x)$, with z as in (2.1), is also a solution to (1.1). Similarly, if $\psi(k,x)$ is a solution to (1.1), then $\psi(z^2k,x)$ is also a solution to (1.1). Even though both $\psi(k,x)$ and $\psi(zk,x)$ are solutions to (1.1), their k-domains do not coincide. In fact, the k-domain of $\psi(zk,x)$ is obtained from the k-domain of $\psi(k,x)$ by rotating the latter k-domain clockwise around the origin of the complex k-plane by $2\pi/3$, i.e. by 120°. For example, we know that the k-domain of the Jost solution $\overline{f}(k,x)$ to (2.49) is $\overline{\Omega}_1$. The k-domain of $\overline{f}(zk,x)$ is obtained from $\overline{\Omega}_1$ by rotating it clockwise by $2\pi/3$, which is illustrated in the left plot of Figure 2.4.

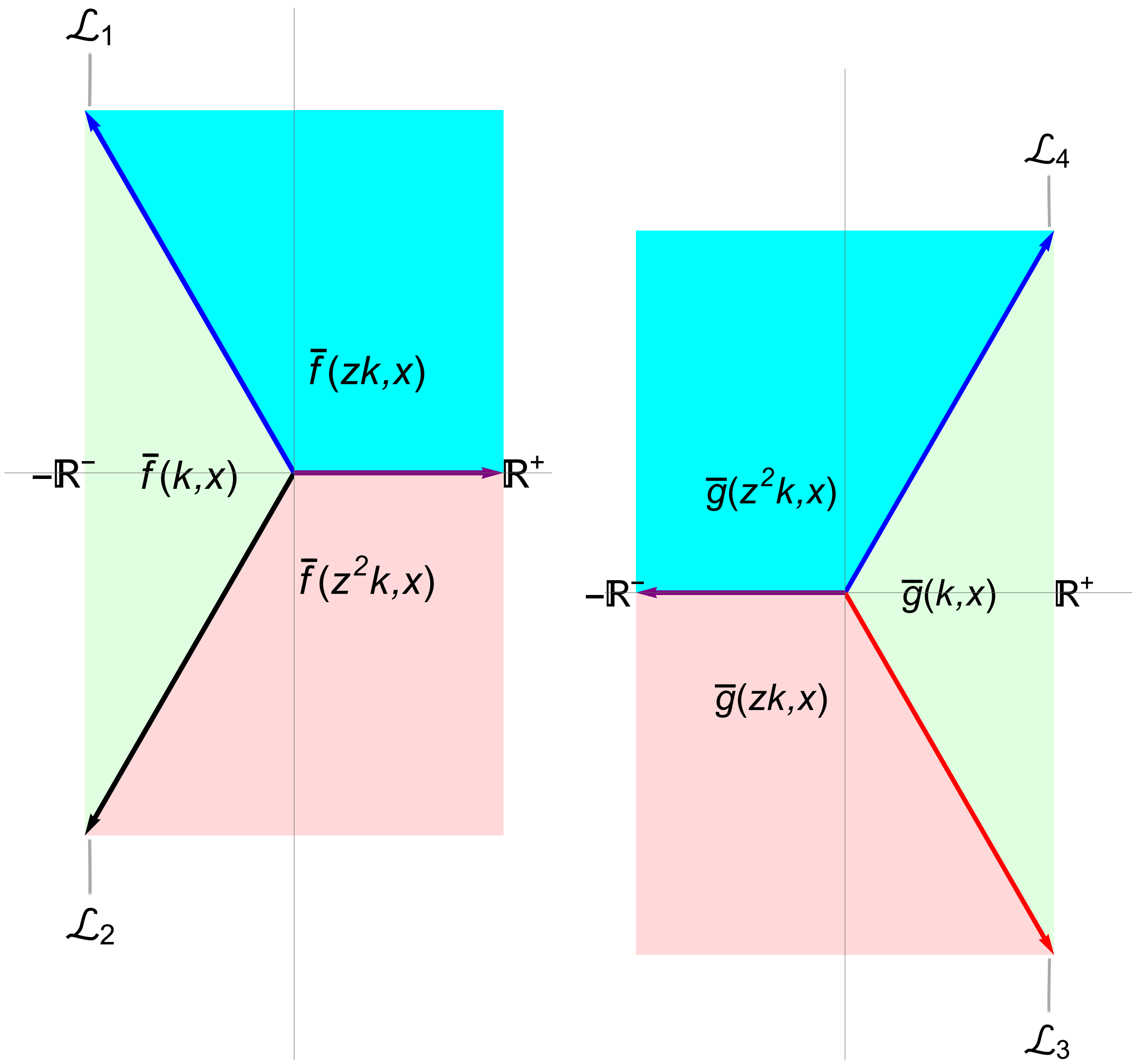

Figure 2.4: The k-domains of $\overline{f}(k,x), \overline{f}(zk,x), \overline{f}(z^2k,x)$, respectively, shown on the left and the k-domains of $\overline{g}(k,x)$, $\overline{g}(zk,x)$, $\overline{g}(z^2k,x)$, respectively, shown on the right.

Since $\psi(z^2k,x)$ is obtained from $\psi(zk,x)$ by replacing k in the latter quantity by zk, the k-domain of $\psi(z^2k,x)$ is obtained from the k-domain of $\psi(zk,x)$ by rotating the latter domain clockwise by $2\pi/3$. Equivalently, we see that the k-domain of $\psi(z^2k,x)$ is obtained from the k-domain of $\psi(k,x)$ by rotating the latter k-domain counterclockwise by $2\pi/3$. In particular, the k-domain of $\overline{f}(z^2k,x)$, which can be obtained from the k-domain of $\overline{f}(k,x)$ or $\overline{f}(zk,x)$ as we have explained, is illustrated in the left plot of Figure 2.4. In a similar manner, the k-domains of $\overline{g}(zk,x)$ and $\overline{g}(z^2k,x)$ are obtained from the k-domain of the right Jost solution $\overline{g}(k,x)$, as illustrated in the right plot of Figure 2.4. Since $\overline{f}(k,x)$ and $\overline{g}(k,x)$ are the Jost solutions associated with the adjoint equation (2.49), we conclude that $\overline{f}(zk,x)$, $\overline{g}(zk,x)$, $\overline{f}(z^2k,x)$, and $\overline{g}(z^2k,x)$ are also solutions to (2.49).

solution	**k-domain**	**boundaries**
$\overline{f}(k,x)$	$\overline{\Omega}_1^{\text{up}} \cup \overline{\Omega}_1^{\text{down}}$	$\mathcal{L}_1$, $\mathcal{L}_2$
$\overline{f}(zk,x)$	$\overline{\Omega}_3^{\text{up}} \cup \overline{\Omega}_4$	$\mathbb{R}^+$, $\mathcal{L}_1$
$\overline{f}(z^2k,x)$	$\overline{\Omega}_2 \cup \overline{\Omega}_3^{\text{down}}$	$\mathcal{L}_2$, $\mathbb{R}^+$
$\overline{g}(k,x)$	$\overline{\Omega}_3^{\text{down}} \cup \overline{\Omega}_3^{\text{up}}$	$\mathcal{L}_3$, $\mathcal{L}_4$
$\overline{g}(zk,x)$	$\overline{\Omega}_1^{\text{down}} \cup \overline{\Omega}_2$	$-\mathbb{R}^-$, $\mathcal{L}_3$
$\overline{g}(z^2k,x)$	$\overline{\Omega}_1^{\text{up}} \cup \overline{\Omega}_4$	$\mathcal{L}_4$, $-\mathbb{R}^-$

Table 2.1: The k-domains of the Jost solutions to the adjoint equation (2.49).

To describe the k-domains of each of $\overline{f}(k,x)$, $\overline{g}(k,x)$, $\overline{f}(zk,x)$, $\overline{g}(zk,x)$, $\overline{f}(z^2k,x)$, and $\overline{g}(z^2k,x)$, we divide the complex k-plane into six open sectors by using the directed half lines $\mathcal{L}_1$, $-\mathbb{R}^-$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathbb{R}^+$, and $\mathcal{L}_4$, as indicated in the left plot in Figure 2.5. Those six sectors are denoted by Ω_1^{up}, Ω_1^{down}, Ω_2, Ω_3^{down}, Ω_3^{up}, and Ω_4, respectively, as shown in the left plot in Figure 2.5. Recall that the directed half lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ and the sectors Ω_1, Ω_2, Ω_3, Ω_4 are all described in Figure 2.1. Note that $\mathbb{R}^+$ corresponds to the positive part of the directed real axis, and $-\mathbb{R}^-$ corresponds to the negative part of the real axis directed from $k=0$ to $k=-\infty$. The k-domains of $\overline{f}(k,x)$, $\overline{g}(k,x)$, $\overline{f}(zk,x)$, $\overline{g}(zk,x)$, $\overline{f}(z^2k,x)$, $\overline{g}(z^2k,x)$ are listed in Table 2.1, along with their respective boundaries. The right plot in

Figure 2.5 helps illustrate the contents of Table 2.1 and provides easy referencing for the k-domains of $\overline{f}(k,x)$, $\overline{g}(k,x)$, $\overline{f}(zk,x)$, $\overline{g}(zk,x)$, $\overline{f}(z^2k,x)$, $\overline{g}(z^2k,x)$.

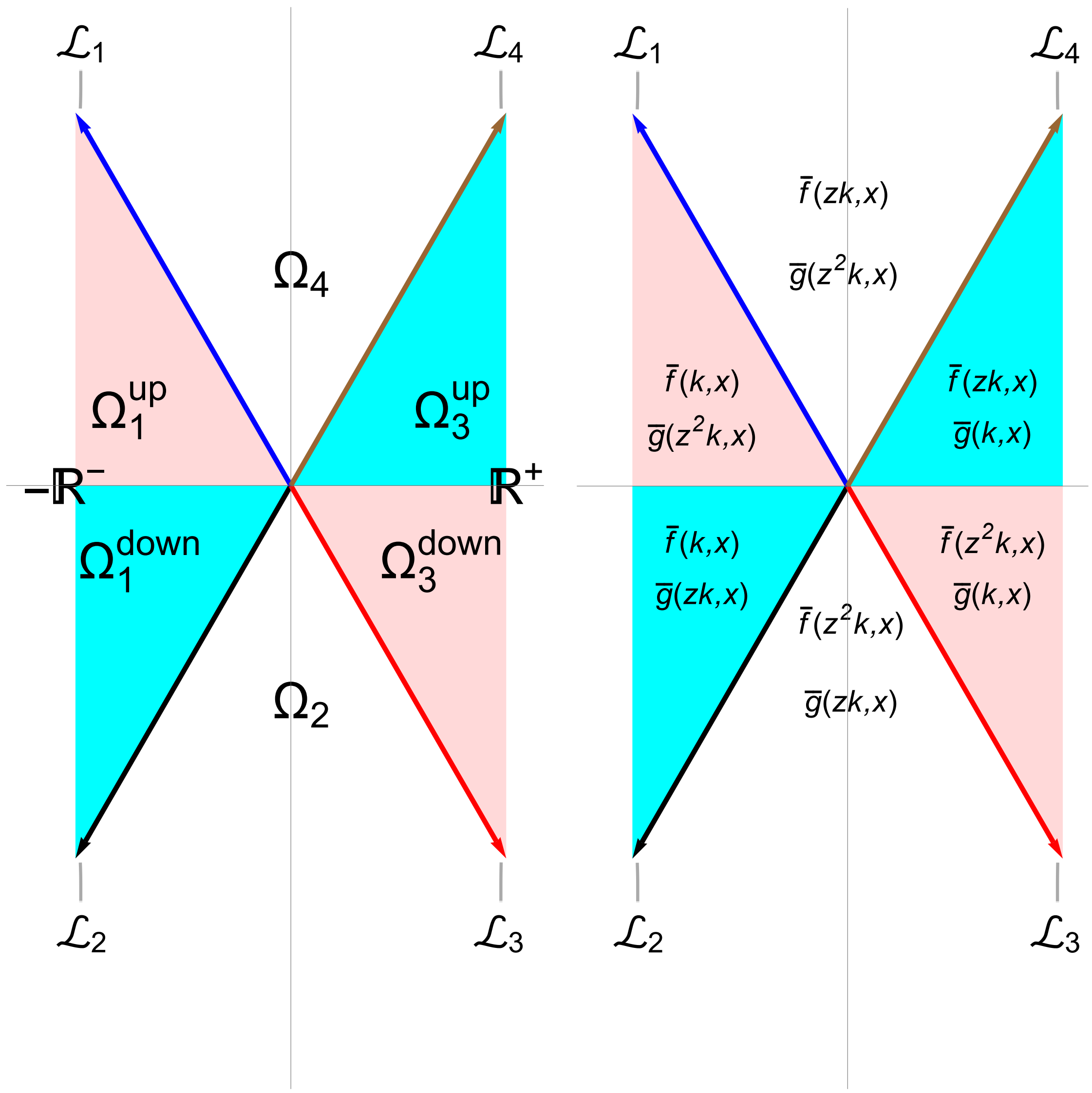

Figure 2.5: The complex k-plane divided into the six sectors Ω_1^{up}, Ω_1^{down}, Ω_2, Ω_3^{down}, Ω_3^{up}, and Ω_4 as shown on the left, and the k-domains of $\overline{f}(k,x)$, $\overline{g}(k,x)$, $\overline{f}(zk,x)$, $\overline{g}(zk,x)$, $\overline{f}(z^2k,x)$, $\overline{g}(z^2k,x)$, respectively, shown on the right.

2.8 The 2-Wronskian and the 3-Wronskian

Having presented the adjoint equation (2.49), the Jost solutions $\overline{f}(k,x)$ and $\overline{g}(k,x)$, and the k-domains of those two Jost solutions, we next introduce the 2-Wronskian and the 3-Wronskian, respectively. We define the 2-Wronskian $[F(x)\,;G(x)]$ of any two differentiable functions $F(x)$ and $G(x)$ as

$$[F(x)\,;G(x)] := \begin{vmatrix} F(x) & G(x) \\ F'(x) & G'(x) \end{vmatrix}, \tag{2.53}$$

where the absolute bars denote the 2×2 matrix determinant and we recall that the prime denotes the x-derivative. We also define the 3-Wronskian of any three differentiable functions $F(x)$, $G(x)$, and $H(x)$ in terms of the 3×3 matrix determinant as

$$[F(x);\, G(x);\, H(x)] := \begin{vmatrix} F(x) & G(x) & H(x) \\ F'(x) & G'(x) & H'(x) \\ F''(x) & G''(x) & H''(x) \end{vmatrix}. \tag{2.54}$$

We remark that (2.53) is equivalent to

$$[F(x)\,;G(x)] = F(x)\,G'(x) - F'(x)\,G(x). \tag{2.55}$$

At times, the 2-Wronskian of $F(x)$ and $G(x)$ defined by other authors may differ by a minus sign from our definition in (2.53). We prefer to define the 2-Wronskian as in (2.53) so that the concept of Wronskian can easily be generalized from (2.53) and (2.54) by defining the n-Wronskian of n functions of x with the help of the $n\times n$ matrix determinant.

In the next theorem, we construct a solution to (1.1) in terms of the 2-Wronskian of two solutions to the adjoint equation (2.49).

Theorem 2.8.1. *Let $\overline{\psi}(k,x)$ and $\overline{\phi}(k,x)$ be two solutions to the adjoint equation (2.49) with their respective k-domains that do not necessarily coincide. Then, the 2-Wronskian given by $\left[\overline{\psi}(-zk^*,x)^*\,;\,\overline{\phi}(-z^2k^*,x)^*\right]$ is a solution to (1.1), where the k-domain of the 2-Wronskian is determined by the intersection of the respective k-domains of $\overline{\psi}(-zk^*,x)^*$ and $\overline{\phi}(-z^2k^*,x)^*$.*

Proof. Since $\overline{\psi}(k,x)$ and $\overline{\phi}(k,x)$ each satisfy the adjoint equation (2.49), we first obtain the two ODEs satisfied by $\overline{\psi}(-zk^*,x)^*$ and $\overline{\phi}(-z^2k^*,x)^*$, respectively. Then, we form the 2-Wronskian $\left[\overline{\psi}(-zk^*,x)^*\,;\,\overline{\phi}(-z^2k^*,x)^*\right]$ by using (2.55), which yields

$$\left[\overline{\psi}(-zk^*,x)^*\,;\,\overline{\phi}(-z^2k^*,x)^*\right]=\overline{\psi}(-zk^*,x)^*\,\overline{\phi}'(-z^2k^*,x)^*-\overline{\psi}'(-zk^*,x)^*\,\overline{\phi}(-z^2k^*,x)^*. \tag{2.56}$$

With the help of (2.50) relating the adjoint potentials $\overline{Q}(x)$ and $\overline{P}(x)$ to the potentials $Q(x)$ and $P(x)$, we directly verify that the right-hand side of (2.56) satisfies (1.1). ■

In Theorem 2.8.1, by choosing $\overline{\psi}(k,x)$ as the left Jost solution $\overline{f}(z^2k,x)$ and by choosing $\overline{\phi}(k,x)$ as the right Jost solution $\overline{g}(k,x)$, we see that $\left[\overline{f}(-z^2k^*,x)^*\,;\,\overline{g}(-zk^*,x)^*\right]$ is a solution to (1.1). We use $h^{\text{down}}(k,x)$, to denote that solution, i.e. we let

$$h^{\text{down}}(k,x):=\left[\overline{f}(-z^2k^*,x)^*\,;\,\overline{g}(-zk^*,x)^*\right]. \tag{2.57}$$

Similarly, in Theorem 2.8.1, by choosing $\overline{\psi}(k,x)$ as $\overline{f}(k,x)$ and choosing $\overline{\phi}(k,x)$ as $\overline{g}(z^2k,x)$, we see that $\left[\overline{f}(-zk^*,x)^*\,;\,\overline{g}(-z^2k^*,x)^*\right]$ is a solution to (1.1). We use $h^{\text{up}}(k,x)$ to denote that solution, i.e. we define

$$h^{\text{up}}(k,x):=\left[\overline{f}(-zk^*,x)^*\,;\,\overline{g}(-z^2k^*,x)^*\right]. \tag{2.58}$$

We remark that the k-domains of $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ are obtained from (2.57) and (2.58), respectively, by using the appropriate transformations of the k-domains of $\overline{f}(k,x)$ and $\overline{g}(k,x)$. The uses of the superscripts down and up in $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ are due to the locations of their k-domains $\overline{\Omega}_2$ and $\overline{\Omega}_4$, respectively, in the complex plane $\mathbb{C}$, as we will see in Section 2.9.

Note that the k-domain of the function $\psi(-k,x)$ is obtained from the k-domain of $\psi(k,x)$ by reflecting the latter k-domain through the origin of the complex k-plane. Similarly, the k-domain of the function $\psi(k^*,x)$ is obtained by reflecting the k-domain of $\psi(k,x)$ along the real axis in the complex k-plane. Furthermore, the k-domain of $\psi(k,x)^*$ is the same as the k-domain of $\psi(k,x)$. This is due to the fact that if $\psi(k,x)$ is well defined at some k-value, then $\psi(k,x)^*$ is also well defined as the complex conjugate at that k-value. In Section 2.7, we have already discussed how to obtain the k-domains of $\psi(zk,x)$ and $\psi(z^2k,x)$ from the k-domain of $\psi(k,x)$. Thus, we can obtain the k-domains of $\overline{f}(-zk^*,x)^*$, $\overline{g}(-z^2k^*,x)^*$, $\overline{f}(-z^2k^*,x)^*$, and $\overline{g}(-zk^*,x)^*$ from the k-domains of $\overline{f}(k,x)$ and $\overline{g}(k,x)$ by applying the appropriate transformations in the complex k-plane. This also helps us to determine the k-domains of $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$.

2.9 The k-domains of $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$

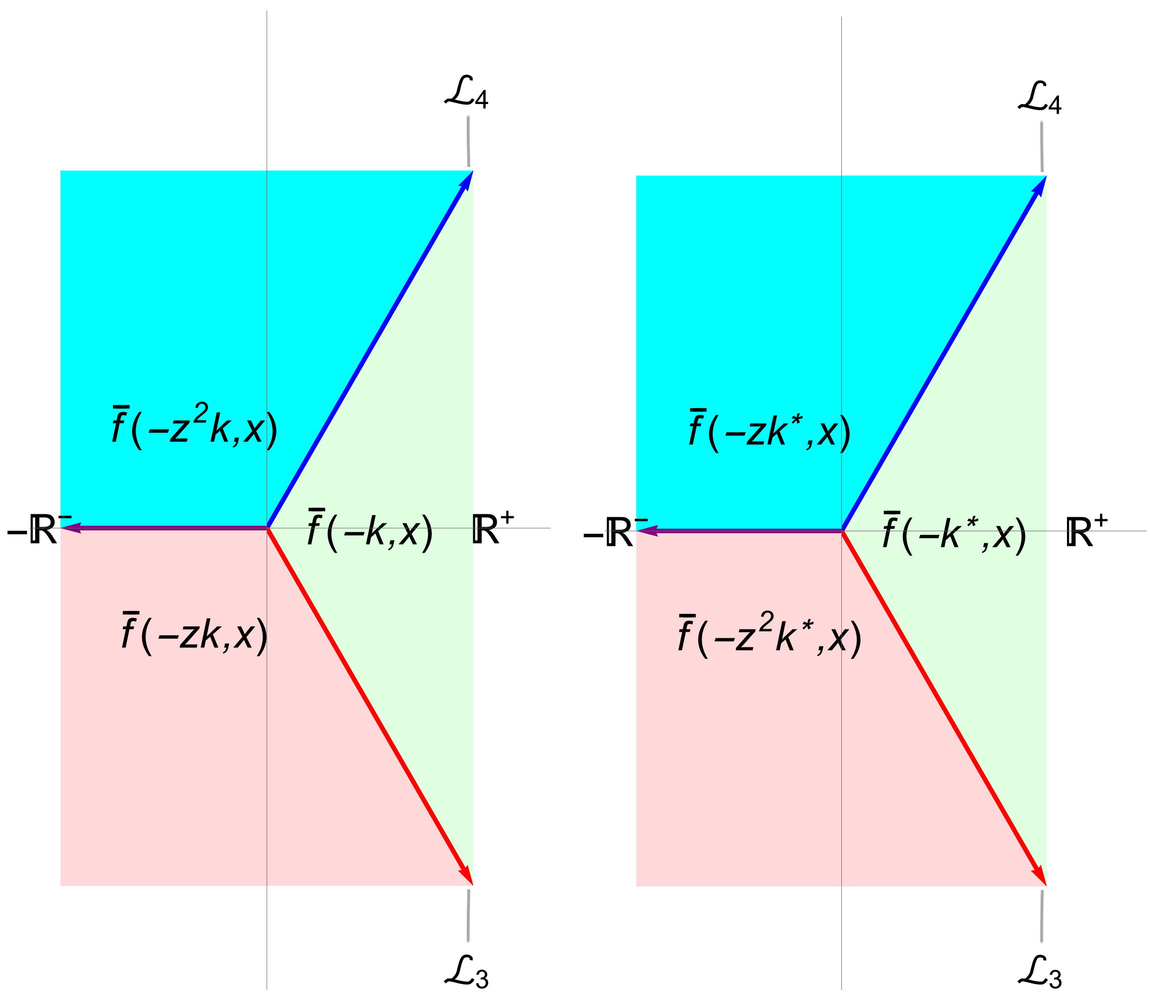

Figure 2.6: The k-domains of $\overline{f}(-k,x)$, $\overline{f}(-zk,x)$, $\overline{f}(-z^2k,x)$, respectively, as shown on the left, and the k-domains of $\overline{f}(-k^*,x)$, $\overline{f}(-zk^*,x)$, $\overline{f}(-z^2k^*,x)$, respectively, on the right.

As mentioned in Section 2.8, we can obtain the k-domains of $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ by using a number of successive transformations on the k-domains of $\overline{f}(k,x)$ and $\overline{g}(k,x)$. Toward this goal, we first establish the k-domain of each of $\overline{f}(-zk^*,x)$, $\overline{f}(-z^2k^*,x)$, $\overline{g}(-zk^*,x)$, and $\overline{g}(-z^2k^*,x)$. Then, to establish the k-domain of $h^{\text{down}}(k,x)$, as seen from (2.57), we use the intersection of the k-domains of $\overline{f}(-z^2k^*,x)$ and $\overline{g}(-zk^*,x)$. Similarly, as seen from (2.58), we obtain the k-domain of $h^{\text{up}}(k,x)$ by using the intersection of the k-domains of $\overline{f}(-zk^*,x)$ and $\overline{g}(-z^2k^*,x)$. To determine the k-domains of $\overline{f}(-zk^*,x)$, $\overline{f}(-z^2k^*,x)$, $\overline{g}(-zk^*,x)$, and $\overline{g}(-z^2k^*,x)$, we proceed as follows. We start with the k-domains of $\overline{f}(k,x)$, $\overline{f}(zk,x)$, and $\overline{f}(z^2k,x)$, as shown in the left plot in Figure 2.4. With the help of reflections with respect to the origin of the complex plane $\mathbb{C}$, we obtain the k-domains of $\overline{f}(-k,x)$, $\overline{f}(-zk,x)$, $\overline{f}(-z^2k,x)$, as seen in the right plot in Figure 2.6. The reflection of a k-domain with respect to the origin is equivalent to replacing k with $-k$ in the arguments of the relevant functions. As seen from the left and right plots in Figure 2.6, by using reflections with respect to the real axis of the complex plane $\mathbb{C}$, we transform the k-domains of $\overline{f}(-k,x)$, $\overline{f}(-zk,x)$ and $\overline{f}(-z^2k,x)$ into the k-domains of $\overline{f}(-k^*,x)$, $\overline{f}(-zk^*,x)$ and $\overline{f}(-z^2k^*,x)$. These transformations amount to replacing k with k^* in the arguments of the relevant functions. Recall that by taking the complex conjugate of a function, we do not change its k-domain. In a similar way, as seen from the right plot in Figure 2.4 and the left plot in 2.7, we obtain the k-domains of $\overline{g}(-k,x)$, $\overline{g}(-zk,x)$, and $\overline{g}(-z^2k,x)$ from the k-domains of $\overline{g}(k,x)$, $\overline{g}(zk,x)$, and $\overline{g}(z^2k,x)$, respectively. This is accomplished by using the appropriate reflections with respect to the origin of the complex k-plane. Similarly, as seen from the left and right plots in Figures 2.7, we obtain the k-domains of $\overline{g}(-k^*,x)$, $\overline{g}(-zk^*,x)$, and $\overline{g}(-z^2k^*,x)$ from the k-domains of $\overline{g}(k,x)$, $\overline{g}(zk,x)$, and $\overline{g}(z^2k,x)$, respectively, by using the appropriate reflection with respect to the real axis on the complex k-plane.

As seen from (2.57), we obtain the k-domain of $h^{\text{down}}(k,x)$ by using the intersection of the k-domains of $\overline{f}(-z^2k^*,x)$ and $\overline{g}(-zk^*,x)$. From the right plot in Figure 2.6 and the right plot in Figure 2.7, we see that that intersection is given by the sector $\overline{\Omega}_2$, where the

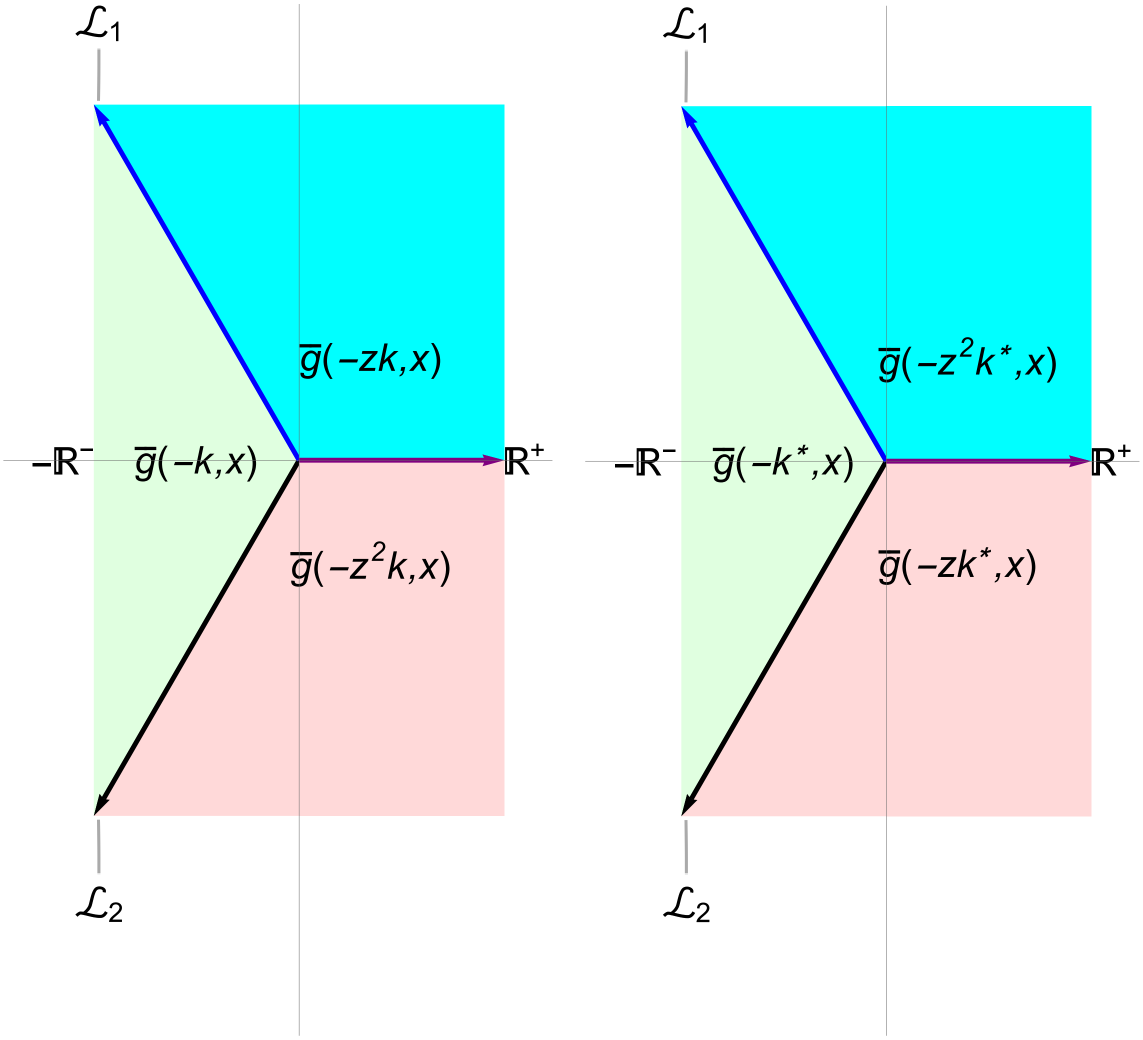

Figure 2.7: The k-domains of $\overline{g}(-k,x)$, $\overline{g}(-zk,x)$, $\overline{g}(-z^2k,x)$, respectively, as shown on the left, and the k-domains of $\overline{g}(-k^*,x)$, $\overline{g}(-zk^*,x)$, $\overline{g}(-z^2k^*,x)$, respectively, on the right.

sector Ω_2 and its boundaries $\mathcal{L}_2$ and $\mathcal{L}_3$ are shown in Figure 2.1. Similarly, from (2.58), we see that the k-domain of $h^{\rm up}(k,x)$ is obtained as the intersection of the k-domains of $\overline{f}(-zk^*,x)$ and $\overline{g}(-z^2k^*,x)$, as seen from the right plot in Figure 2.6 and the right plot in Figure 2.7. That intersection yields the sector $\overline{\Omega}_4$, where the sector Ω_4 and its boundaries $\mathcal{L}_1$ and $\mathcal{L}_4$ are shown in Figure 2.1. In summary, the sector $\overline{\Omega}_2$ is the k-domain of $h^{\rm down}(k,x)$ and the sector $\overline{\Omega}_4$ is the k-domain of $h^{\rm up}(k,x)$.

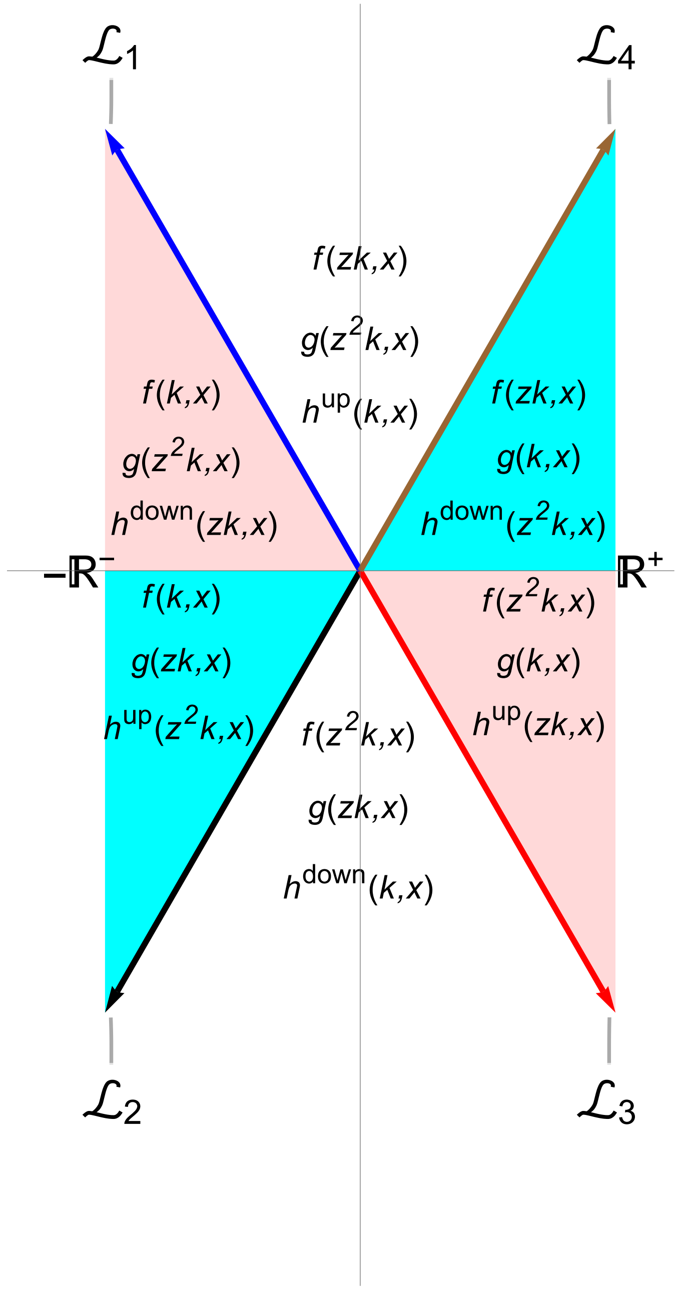

Figure 2.8: The k-domains of $f(k,x)$, $f(zk,x)$, $f(z^2k,x)$, $g(k,x)$, $g(zk,x)$, $g(z^2k,x)$, $h^{\text{down}}(k,x)$, $h^{\text{down}}(zk,x)$, $h^{\text{down}}(z^2k,x)$, $h^{\text{up}}(k,x)$, $h^{\text{up}}(zk,x)$, and $h^{\text{up}}(z^2k,x)$.

In Section 2.4, we have established the k-domains of the Jost solutions $f(k,x)$ and $g(k,x)$ to (1.1) by determining the k-values for which the integral equations (2.25) and (2.33) have solutions represented by uniformly convergent series. This has indicated that

$\overline{\Omega}_1$ and $\overline{\Omega}_3$ are the respective k-domains of $f(k,x)$ and $g(k,x)$. In this section, we have obtained the k-domains of $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ with the help of some appropriate transformations in the complex k-plane. By doing so, we have determined the k-domains of $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ as $\overline{\Omega}_2$ and $\overline{\Omega}_4$, respectively. By proceeding in a similar manner and by exploiting the aforementioned transformations in the complex k-plane, we determine the k-domains of all solutions to (1.1). In particular, we obtain the k-domains of $f(zk,x)$, $f(z^2k,x)$, $g(zk,x)$, $g(z^2k,x)$, $h^{\text{down}}(zk,x)$, $h^{\text{down}}(z^2k,x)$, $h^{\text{up}}(zk,x)$, and $h^{\text{up}}(z^2k,x)$. The relevant k-domains of all the twelve solutions are illustrated in Figure 2.8. The latter eight solutions as well as the four solutions $f(k,x)$, $g(k,x)$, $h^{\text{down}}(k,x)$, and $h^{\text{up}}(k,x)$ enable us to express any solution to (1.1) by using three appropriate linearly independent solutions among those twelve solutions. We choose the three linearly independent solutions by taking into consideration the common k-domains indicated in Figure 2.8. The linear dependence or independence of any three solutions to (1.1) can be established by checking whether the corresponding 3-Wronskian is zero or nonzero, respectively. This will be done in Chapter 3, where we will evaluate and establish the linear independence of the relevant three solutions in each of the six sectors shown in Figure 2.8.

2.10 The analyticity and continuity in k

We refer to $f(k,x)$, $g(k,x)$, $h^{\text{down}}(k,x)$, and $h^{\text{up}}(k,x)$ collectively as the four fundamental solutions for (1.1). This is because the remaining eight solutions $f(zk,x)$, $f(z^2k,x)$, $g(zk,x)$, $g(z^2k,x)$, $h^{\text{down}}(zk,x)$, $h^{\text{down}}(z^2k,x)$, $h^{\text{up}}(zk,x)$, and $h^{\text{up}}(z^2k,x)$ are obtained by replacing k with zk or z^2k in those four fundamental solutions. In the following theorem, we present the analyticity and continuity properties in k for those four solutions. Even though the relevant properties of $f(k,x)$ and $g(k,x)$ are already stated in Corollaries 2.4.2.1 and 2.4.3.1, for the convenience of the reader we restate those properties in our theorem.

Theorem 2.10.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $f(k,x)$, $g(k,x)$, $h^{\text{down}}(k,x)$, and $h^{\text{up}}(k,x)$ be the four fundamental solutions* *to* (1.1) *appearing in* (2.21), (2.29), (2.57), (2.58), *respectively. We recall that* $\overline{\Omega}_1$, $\overline{\Omega}_3$, $\overline{\Omega}_2$, *and* $\overline{\Omega}_4$, *respectively, are the* k*-domains of those four fundamental solutions, as illustrated in Figure 2.1. Then, for each fixed* $x \in \mathbb{R}$, *we have the following:*

(a) *The left Jost solution* $f(k,x)$ *is analytic in* $k \in \Omega_1$ *and continuous in* $k \in \overline{\Omega}_1$.

(b) *The right Jost solution* $g(k,x)$ *is analytic in* $k \in \Omega_3$ *and continuous in* $k \in \overline{\Omega}_3$.

(c) *The solution* $h^{\text{down}}(k,x)$ *is analytic in* $k \in \Omega_2$ *and continuous in* $k \in \overline{\Omega}_2$.

(d) *The solution* $h^{\text{up}}(k,x)$ *is analytic in* $k \in \Omega_4$ *and continuous in* $k \in \overline{\Omega}_4$.

Proof. As already indicated, the result in (a) is given in Corollary 2.4.2.1(b) and that the result in (b) is given in Corollary 2.4.3.1(b). The proof of (c) is obtained by using (2.57) and the relevant k-domains of the continuity and analyticity of the adjoint quantities appearing on the right-hand side of (2.57). In a similar manner, we obtain the proof of (d) by using (2.58) and the relevant k-domains of the continuity and analyticity of the adjoint quantities appearing on the right-hand side of (2.58). ■

We recall that, as a result of (2.50), the adjoint potentials $\overline{Q}(x)$ and $\overline{P}(x)$ in (2.49) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ whenever the potentials $Q(x)$ and $P(x)$ in (1.1) are in $\mathcal{S}(\mathbb{R})$. Hence, the properties of each of the four fundamental solutions to (2.49) are similar to the properties of the corresponding fundamental solution to (1.1). For example, the solution $\overline{f}(k,x)$ to (2.49) and the solution $f(k,x)$ to (1.1) have similar analyticity and continuity properties in k. The same holds for the remaining fundamental solutions to (2.49) and (1.1), respectively. Thus, Theorem 2.10.1 yields the following corollary.

Corollary 2.10.1.1. *Assume that the potentials* $Q(x)$ *and* $P(x)$ *in* (1.1) *belong to the Schwartz class* $\mathcal{S}(\mathbb{R})$. *Let* $\overline{f}(k,x)$ *and* $\overline{g}(k,x)$ *be the Jost solutions to the adjoint equation* (2.49), *appearing in* (2.51) *and* (2.52), *respectively. Let* $\overline{h}^{\text{down}}(k,x)$ *and* $\overline{h}^{\text{up}}(k,x)$ *be the solutions to the adjoint equation* (2.49), *which are defined in terms of the Jost solutions to* (1.1) *as*

$$\overline{h}^{\text{down}}(k,x) := \left[f(-z^2k^*,x)^*\,;\,g(-zk^*,x)^*\right], \tag{2.59}$$

$$\overline{h}^{\text{up}}(k,x) := \Big[f(-zk^*,x)^*\,;\ g(-z^2k^*,x)^* \Big]. \tag{2.60}$$

Then, for each fixed $x \in \mathbb{R}$, we have the following:

(a) *The left Jost solution $\overline{f}(k,x)$ is analytic in $k \in \Omega_1$ and continuous in $k \in \overline{\Omega}_1$.*

(b) *The right Jost solution $\overline{g}(k,x)$ is analytic in $k \in \Omega_3$ and continuous in $k \in \overline{\Omega}_3$.*

(c) *The solution $\overline{h}^{\text{down}}(k,x)$ is analytic in $k \in \Omega_2$ and continuous in $k \in \overline{\Omega}_2$.*

(d) *The solution $\overline{h}^{\text{up}}(k,x)$ is analytic in $k \in \Omega_4$ and continuous in $k \in \overline{\Omega}_4$.*

In the next theorem, we present the analyticity and continuity properties in k for the scattering coefficients for (1.1).

Theorem 2.10.2. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $T_{\text{l}}(k)$, $L(k)$, and $M(k)$ be the left scattering coefficients appearing in (2.35) and similarly let $T_{\text{r}}(k)$, $R(k)$, and $N(k)$ be the right scattering coefficients appearing in (2.39). We have the following:*

(a) *The reciprocal $T_{\text{l}}(k)^{-1}$ of the left transmission coefficient $T_{\text{l}}(k)$ is analytic in Ω_1 and continuous in $\overline{\Omega}_1 \setminus \{0\}$.*

(b) *The reciprocal $T_{\text{r}}(k)^{-1}$ of the right transmission coefficient $T_{\text{r}}(k)$ is analytic in Ω_3 and continuous in $\overline{\Omega}_3 \setminus \{0\}$.*

(c) *The quantity $L(k)\,T_{\text{l}}(k)^{-1}$ is continuous on the directed half line $\mathcal{L}_1 \setminus \{0\}$.*

(d) *The quantity $M(k)\,T_{\text{l}}(k)^{-1}$ is continuous on the directed half line $\mathcal{L}_2 \setminus \{0\}$.*

(e) *The quantity $R(k)\,T_{\text{r}}(k)^{-1}$ is continuous on the directed half line $\mathcal{L}_3 \setminus \{0\}$.*

(f) *The quantity $N(k)\,T_{\text{r}}(k)^{-1}$ is continuous on the directed half line $\mathcal{L}_4 \setminus \{0\}$.*

Proof. The proof of (a) is obtained by using the integral representation for $T_{\text{l}}(k)^{-1}$ given in (2.36) and by using the k-domains of the continuity and analyticity of $u(k,x)$ established in Theorem 2.4.2(c). Similarly, the proof of (b) is obtained with the help of the integral representation for $T_{\text{r}}(k)^{-1}$ given in (2.40) and by using the k-domains of the continuity and analyticity of $v(k,x)$ established in Theorem 2.4.3(c). The proofs of (c)–(f) are established in a similar manner by using the integral representations for $L(k)\,T_{\text{l}}(k)^{-1}$, $M(k)\,T_{\text{l}}(k)^{-1}$, $R(k)\,T_{\text{r}}(k)^{-1}$, $N(k)\,T_{\text{r}}(k)^{-1}$ given in (2.37), (2.38), (2.41), (2.42), respectively, and by using the k-domains of the continuity of $u(k,x)$ and $v(k,x)$ stated in Theorems 2.4.2(c) and 2.4.3(c), respectively. ■

CHAPTER 3

THE DIRECT SCATTERING PROBLEM: PART II

In Chapter 2, we have described the direct scattering problem for (1.1). We have also introduced the left Jost solution $f(k, x)$ and the right Jost solution $g(k, x)$ to (1.1) along with their respective k-domains and spacial asymptotics. In addition to the Jost solutions $f(k, x)$ and $g(k, x)$, by introducing the 2-Wronskian and the adjoint equation (2.49), we have constructed two additional solutions $h^{\text{down}}(k, x)$ and $h^{\text{up}}(k, x)$ to (1.1), and we have determined their respective k-domains.

We recall that, at any fixed k-value in $\mathbb{C}$, the 3-Wronskian of any three solutions to (1.1) is either identically zero for all $x \in \mathbb{R}$ or never becomes zero at any x-value in $\mathbb{R}$. The former happens when those three solutions are linearly dependent at that k-value, and the latter occurs when the three solutions are linearly independent at that k-value. Since the coefficient of the term ψ'' in (1.1) is zero, the 3-Wronskian of any three solutions to (1.1) is independent of x, and hence the value of that 3-Wronskian can be evaluated at any x-value, including when $x \to \pm\infty$. This x-independence allows us to relate the scattering coefficients for (1.1) to the 3-Wronskians of certain particular solutions to (1.1).

Through the use of the 3-Wronskian, we first establish the linear dependence of three certain solutions to (1.1) on each of the respective boundaries $\mathcal{L}_1$, $-\mathbb{R}^-$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathbb{R}^+$, and $\mathcal{L}_4$. Then, the linear dependence of that set of three solutions on each of six directed boundaries allows us to formulate various relationships among the primary and secondary reflection coefficients for the adjoint equation (2.49) and the primary and secondary reflection coefficients for (1.1). The linear independence of any three solutions to (1.1) via the 3-Wronskian on the six sectors as illustrated in Figure 2.5, then establishes various relationships between the transmission coefficients for (1.1) and (2.49), respectively.

In this chapter, we introduce the scattering coefficients for the adjoint equation (2.49) and determine the spacial asymptotics of the solutions $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ to (1.1) appearing in (2.57) and (2.58), respectively. We also establish the relationships among the scattering coefficients for (1.1) and the scattering coefficients for the adjoint equation (2.49). We then present the large k-asymptotics of the four fundamental solutions $f(k,x)$, $g(k,x)$, $h^{\text{down}}(k,x)$, and $h^{\text{up}}(k,x)$ to (1.1). We also determine the large and small k-asymptotics of the six scattering coefficients for (1.1). Finally, in this chapter, we provide the basic relevant information on the bound states of (1.1) and introduce the concept of a dependency constant at each bound state.

3.1 The scattering coefficients for the adjoint equation (2.49)

As mentioned in Chapter 2, the left Jost solutions $f(k,x)$ to (1.1) and $\overline{f}(k,x)$ to (2.49), respectively, have similar properties. Also, the respective right Jost solutions $g(k,x)$ and $\overline{g}(k,x)$ have similar properties. Consequently, the scattering coefficients for the adjoint equation (2.49) are defined in a similar manner, and this is done by using the spacial asymptotics of the Jost solutions $\overline{f}(k,x)$ and $\overline{g}(k,x)$ to (2.49).

For the adjoint equation (2.49), we introduce the left and right transmission coefficients $\overline{T}_{\rm l}(k)$ and $\overline{T}_{\rm r}(k)$, respectively, the left and right primary reflection coefficients $\overline{L}(k)$ and $\overline{R}(k)$, respectively, and the left and right secondary reflection coefficients $\overline{M}(k)$ and $\overline{N}(k)$, respectively. This is done by using the analogs of (2.35)–(2.38) for the left scattering coefficients for (1.1) and using the analogs of (2.39)–(2.42) for the right scattering coefficients for (1.1). As $x \to -\infty$, the spacial asymptotics of the left Jost solution $\overline{f}(k,x)$ to (2.49) is given as

$$\overline{f}(k,x)=\begin{cases} e^{kx}\,\overline{T}_{\rm l}(k)^{-1}[1+o(1)]+e^{zkx}\,\overline{L}(k)\,\overline{T}_{\rm l}(k)^{-1}[1+o(1)], \qquad k\in\mathcal{L}_1,\\ \\ e^{kx}\,\overline{T}_{\rm l}(k)^{-1}[1+o(1)], \qquad k\in\Omega_1,\\ \\ e^{kx}\,\overline{T}_{\rm l}(k)^{-1}[1+o(1)]+e^{z^2kx}\,\overline{M}(k)\,\overline{T}_{\rm l}(k)^{-1}[1+o(1)], \qquad k\in\mathcal{L}_2,\end{cases} \tag{3.1}$$

which is the counterpart of (2.35). As (2.36)–(2.38) are obtained from (2.35), we get

$$\overline{T}_{\rm l}(k)^{-1} := 1 + \int_{-\infty}^{\infty} dy \left[-\frac{Q(y)^*}{3k} + \frac{P(y)^*}{3k^2} \right] \overline{u}(k,y), \qquad k \in \overline{\Omega_1} \setminus \{0\}, \tag{3.2}$$

$$\overline{L}(k)\,\overline{T}_{\rm l}(k)^{-1} := \int_{-\infty}^{\infty} dy\, e^{i\sqrt{3}z^2ky} \left[-\frac{z}{3k^2} P(y)^* + \frac{z^2}{3k} Q(y)^* \right] \overline{u}(k,y), \qquad k \in \mathcal{L}_1 \setminus \{0\}, \tag{3.3}$$

$$\overline{M}(k)\,\overline{T}_{\rm l}(k)^{-1} := \int_{-\infty}^{\infty} dy\, e^{-i\sqrt{3}zky} \left[-\frac{z^2}{3k^2} P(y)^* + \frac{z}{3k} Q(y)^* \right] \overline{u}(k,y), \qquad k \in \mathcal{L}_2 \setminus \{0\}, \tag{3.4}$$

where we have replaced the adjoint potentials $\overline{Q}(x)$ and $\overline{P}(x)$ by their equivalents in terms of $Q(x)$ and $P(x)$ given in (2.50). We recall that the asterisk denotes complex conjugation. Similarly as $x \to +\infty$, we have the spacial asymptotics of the right Jost solution $\overline{g}(k,x)$ to (2.49) as

$$\overline{g}(k,x) = \begin{cases} e^{kx}\,\overline{T}_{\rm r}(k)^{-1}[1+o(1)] + e^{zkx}\,\overline{R}(k)\,\overline{T}_{\rm r}(k)^{-1}[1+o(1)], & k \in \mathcal{L}_3, \\ e^{kx}\,\overline{T}_{\rm r}(k)^{-1}[1+o(1)], & k \in \Omega_3, \\ e^{kx}\,\overline{T}_{\rm r}(k)^{-1}[1+o(1)] + e^{z^2kx}\,\overline{N}(k)\,\overline{T}_{\rm r}(k)^{-1}[1+o(1)], & k \in \mathcal{L}_4, \end{cases} \tag{3.5}$$

which is the counterpart of (2.39). As (2.39)–(2.42) are obtained from (2.38), we get

$$\overline{T}_{\rm r}(k)^{-1} := 1 + \int_{-\infty}^{\infty} dy \left[\frac{Q(y)^*}{3k} - \frac{P(y)^*}{3k^2} \right] \overline{v}(k,y), \qquad k \in \overline{\Omega_1} \setminus \{0\}, \tag{3.6}$$

$$\overline{R}(k)\,\overline{T}_{\rm r}(k)^{-1} := \int_{-\infty}^{\infty} dy\, e^{i\sqrt{3}z^2ky} \left[\frac{z}{3k^2} P(y)^* - \frac{z^2}{3k} Q(y)^* \right] \overline{v}(k,y), \qquad k \in \mathcal{L}_3 \setminus \{0\}, \tag{3.7}$$

$$\overline{N}(k)\,\overline{T}_{\rm r}(k)^{-1} := \int_{-\infty}^{\infty} dy\, e^{-i\sqrt{3}zky} \left[\frac{z^2}{3k^2} P(y)^* - \frac{z}{3k} Q(y)^* \right] \overline{v}(k,y), \qquad k \in \mathcal{L}_4 \setminus \{0\}. \tag{3.8}$$

We refer collectively to $\overline{T}_{\rm l}(k)$, $\overline{L}(k)$, and $\overline{M}(k)$ as the left scattering coefficients for (2.49) and refer collectively to $\overline{T}_{\rm r}(k)$, $\overline{R}(k)$, and $\overline{N}(k)$ as the right scattering coefficients for (2.49).

3.2 The spacial asymptotics of $h^{\rm down}(k,x)$ and $h^{\rm up}(k,x)$

We recall that $f(k,x)$, $g(k,x)$, $h^{\rm down}(k,x)$, and $h^{\rm up}(k,x)$ are the four fundamental solutions to (1.1). In Theorem 2.5.1, we have presented the spacial asymptotics of $f(k,x)$

for $k \in \overline{\Omega}_1$. We also have presented the spacial asymptotics of $g(k,x)$ for $k \in \overline{\Omega}_3$ in Theorem 2.5.2. In the next two theorems, we present the spacial asymptotics of $h^{\text{down}}(k,x)$ for $k \in \overline{\Omega}_2$ and the spacial asymptotics of $h^{\text{up}}(k,x)$ for $k \in \overline{\Omega}_4$.

Theorem 3.2.1. *Assume that the potentials $Q(x)$ and $P(x)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $h^{\text{down}}(k,x)$ be the solution to (1.1) given in (2.59). Then, we have the following:*

(a) *For $k \in \mathcal{L}_2$, using the parametrization $k = z^2 s$ for $s \geq 0$, we have the spacial asymptotics of $h^{\text{down}}(k,x)$ given by*

$$h^{\text{down}}(z^2 s, x) = \begin{cases} s(1-z)\, e^{z^2 sx} \left[\overline{T}_{\text{r}}(-z^2 s)^*\right]^{-1} \\ \quad + s(1-z^2)\, e^{zsx}\, \overline{N}(-z^2 s)^* \left[\overline{T}_{\text{r}}(-z^2 s)^*\right]^{-1} + o(1), \\ s(1-z)\, e^{z^2 sx} \left[\overline{T}_{\text{l}}(-s)^*\right]^{-1} + o(1), \end{cases} \tag{3.9}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$. We can write (3.9) in terms of k and k^*, for $k \in \mathcal{L}_2$, as

$$h^{\text{down}}(k,x) = \begin{cases} z(1-z)k\, e^{kx} \left[\overline{T}_{\text{r}}(-zk^*)^*\right]^{-1} \\ \quad - (1-z)k\, e^{z^2 kx} \overline{N}(-zk^*)^* \left[\overline{T}_{\text{r}}(-zk^*)^*\right]^{-1} + o(1), \\ z(1-z)k\, e^{kx} \left[\overline{T}_{\text{l}}(-z^2 k^*)^*\right]^{-1} + o(1), \end{cases} \tag{3.10}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$.

(b) *For $k \in \mathcal{L}_3$, using the parametrization $k = -zs$ for $s \geq 0$, we have the spacial asymptotics of $h^{\text{down}}(k,x)$ given by*

$$h^{\text{down}}(-zs, x) = \begin{cases} s(1-z^2)\, e^{-zsx} \left[\overline{T}_{\text{r}}(s)^*\right]^{-1} + o(1), \\ s(1-z^2)\, e^{-zsx} \left[\overline{T}_{\text{l}}(zs)^*\right]^{-1} \\ \quad + s(1-z)\, e^{-z^2 sx}\, \overline{L}(zs)^* \left[\overline{T}_{\text{l}}(zs)^*\right]^{-1} + o(1), \end{cases} \tag{3.11}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$. We can write (3.11) in terms of k and k^*, for $k \in \mathcal{L}_3$, as

$$h^{\text{down}}(k,x) = \begin{cases} z(1-z)k\, e^{kx} \left[\overline{T}_{\text{r}}(-zk^*)^*\right]^{-1} + o(1), \\ z(1-z)k\, e^{kx} \left[\overline{T}_{\text{l}}(-z^2k^*)^*\right]^{-1} \\ \quad + (1-z^2)k\, e^{zkx}\, \overline{L}(-z^2k^*)^* \left[\overline{T}_{\text{l}}(-z^2k^*)^*\right]^{-1} + o(1), \end{cases} \tag{3.12}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$.

(c) *For $k \in \Omega_2$, the spacial asymptotics of $h^{\text{down}}(k,x)$ are given by*

$$h^{\text{down}}(k,x) = \begin{cases} z(1-z)k\, e^{kx} \left[\overline{T}_{\text{r}}(-zk^*)^*\right]^{-1} + o(1), & x \to +\infty, \\ z(1-z)k\, e^{kx} \left[\overline{T}_{\text{l}}(-z^2k^*)^*\right]^{-1} + o(1), & x \to -\infty. \end{cases} \tag{3.13}$$

Proof. The spacial asymptotics listed in (3.9)–(3.13) are obtained directly by using the definition of $h^{\text{down}}(k,x)$ given in (2.57) and with the help of the spacial asymptotics of $\overline{f}(k,x)$ given in (2.51) and (3.1) and the spacial asymptotics of $\overline{g}(k,x)$ given in (2.52) and (3.5). ■

The spacial asymptotics of the solution $h^{\text{up}}(k,x)$ to (1.1) are presented in the next theorem.

Theorem 3.2.2. *Assume that $Q(x)$ and $P(x)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $h^{\text{up}}(k,x)$ be the solution to (1.1) given in (2.60). Then, we have the following:*

(a) *For $k \in \mathcal{L}_1$, using the parametrization $k = zs$ for $s \ge 0$, we have the spacial asymptotics of $h^{\text{up}}(k,x)$ given by*

$$h^{\text{up}}(zs,x) = \begin{cases} s(1-z^2)\, e^{zsx} \left[\overline{T}_{\text{r}}(-zs)^*\right]^{-1} \\ \quad + s(1-z)\, e^{z^2sx} \overline{R}(-zs)^*\, \overline{T}_{\text{r}}(-zs)^{*-1} + o(1), \\ s(1-z^2)\, e^{zsx} \left[\overline{T}_{\text{l}}(-s)^*\right]^{-1} + o(1), \end{cases} \tag{3.14}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$. We can write (3.14) in terms of k and k^*, for $k \in \mathcal{L}_1$, as

$$h^{\mathrm{up}}(k,x) = \begin{cases} -z(1-z)k\, e^{kx} \left[\overline{T}_{\mathrm{r}}(-z^2k^*)^*\right]^{-1} \\ \quad -(1-z^2)k\, e^{zkx}\overline{R}(-z^2k^*)^* \left[\overline{T}_{\mathrm{r}}(-z^2k^*)^*\right]^{-1} + o(1), \\ -z(1-z)k\, e^{kx} \left[\overline{T}_{\mathrm{l}}(-zk^*)^*\right]^{-1} + o(1), \end{cases} \tag{3.15}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$.

(b) *For $k \in \mathcal{L}_4$, using the parametrization $k = -z^2 s$ for $s \geq 0$, we have the spacial asymptotics of $h^{\mathrm{up}}(k,x)$ given by*

$$h^{\mathrm{up}}(-z^2s,x) = \begin{cases} s(1-z)\, e^{-z^2sx} \left[\overline{T}_{\mathrm{r}}(s)^*\right]^{-1} + o(1), \\ s(1-z)\, e^{-z^2sx} \left[\overline{T}_{\mathrm{l}}(z^2s)^*\right]^{-1} \\ \quad + s(1-z^2)\, e^{-zsx}\, \overline{M}(z^2s)^* \left[\overline{T}_{\mathrm{l}}(z^2s)^*\right]^{-1} + o(1), \end{cases} \tag{3.16}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$. We can write (3.16) in terms of k and k^*, for $k \in \mathcal{L}_4$, as

$$h^{\mathrm{up}}(k,x) = \begin{cases} -z(1-z)k\, e^{kx} \left[\overline{T}_{\mathrm{r}}(-z^2k^*)^*\right]^{-1} + o(1), \\ -z(1-z)k\, e^{kx} \left[\overline{T}_{\mathrm{l}}(-zk^*)^*\right]^{-1} \\ \quad + (1-z)k\, e^{z^2kx}\, \overline{M}(-zk^*)^* \left[\overline{T}_{\mathrm{l}}(-zk^*)^*\right]^{-1} + o(1), \end{cases} \tag{3.17}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$.

(c) *For $k \in \Omega_4$, the spacial asymptotics of $h^{\mathrm{up}}(k,x)$ are given by*

$$h^{\mathrm{up}}(k,x) = \begin{cases} -z(1-z)k\, e^{kx} \left[\overline{T}_{\mathrm{r}}(-z^2k^*)^*\right]^{-1} + o(1), & x \to +\infty, \\ -z(1-z)k\, e^{kx} \left[\overline{T}_{\mathrm{l}}(-zk^*)^*\right]^{-1} + o(1), & x \to -\infty. \end{cases} \tag{3.18}$$

Proof. The proof is similar to the proof of Theorem 3.2.1. The spacial asymptotics listed in (3.14)–(3.18) are directly obtained by using the definition of $h^{\rm up}(k,x)$ given in (2.58) and with the help of the spacial asymptotics of $\overline{f}(k,x)$ given in (2.51) and (3.1) and the spacial asymptotics of $\overline{g}(k,x)$ given in (2.52) and (3.5). ■

3.3 The relationships among the scattering coefficients

In this section we provide various relationships involving the scattering coefficients for (1.1) and the adjoint scattering coefficients for (2.49). We obtain those relationships by evaluating the 3-Wronskians of various solutions to (1.1) and by using the fact that those Wronskians are independent of x.

We recall that we have partitioned the complex k-plane into six sectors, as done in Figure 2.5. In the interior of each sector, we have three linearly independent solutions to (1.1). Those six sets of three linearly independent solutions are listed in their respective sectors, as shown in Figure 2.8. For example, for $k \in \Omega_1^{\rm up}$, we have the three linearly independent solutions $f(k,x)$, $h^{\rm down}(zk,x)$, $g(z^2k,x)$. Similarly, for $k \in \Omega_4$, we have the three linearly independent solutions $h^{\rm up}(k,x)$, $f(zk,x)$, $g(z^2k,x)$. The 3-Wronskian of the three linearly independent solutions to (1.1) in each of the six sectors of Figure 2.8 then produces certain relationships involving the scattering coefficients for (1.1) and the adjoint scattering coefficients for (2.49).

On the boundary $\mathcal{L}_1$, separating the sectors $\Omega_1^{\rm up}$ and Ω_4, we have five distinct solutions to (1.1), which yield ten 3-Wronskians. In a similar way, by using the corresponding five solutions to (1.1) on each of the other five boundaries $-\mathbb{R}^-$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathbb{R}^+$, and $\mathcal{L}_4$, we obtain various relationships involving the scattering coefficients for (1.1) and (2.49). Hence, by evaluating the spacial asymptotics of the 3-Wronskians on each of the six open sectors and the ten 3-Wronskians on each of the five boundaries, we find the relevant relationships involving the scattering coefficients for (1.1) and the adjoint scattering coefficients for (2.49).

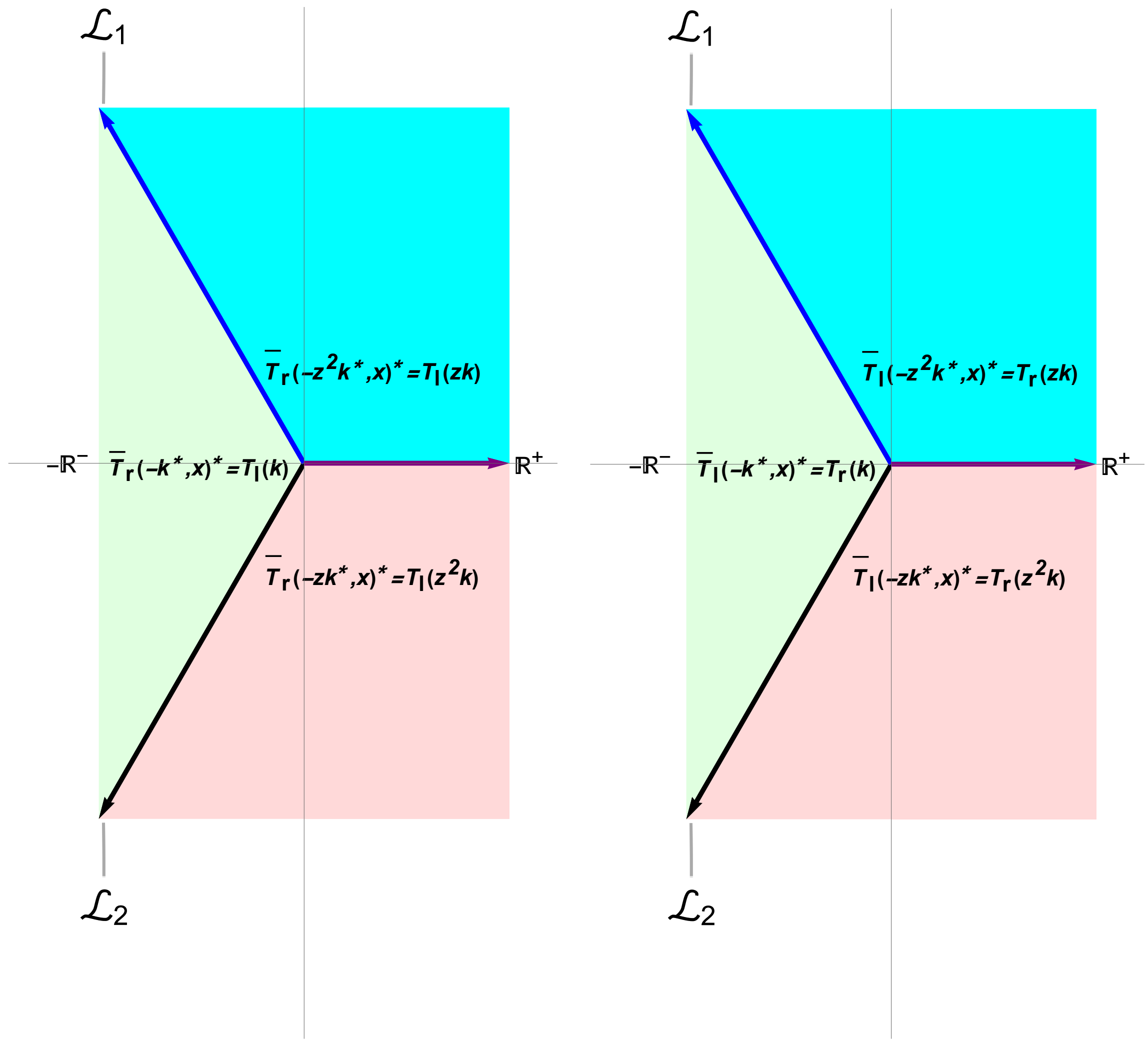

Figure 3.1: The relationships between the adjoint transmission coefficients for (2.49) and the transmission coefficients for (1.1) in the relevant k-domains in $\mathbb{C}$.

In the next theorem, we present the relationships between the adjoint transmission coefficients for (2.49) and the transmission coefficients for (1.1). These relationships are obtained by evaluating the 3-Wronskian of the relevant solutions to (1.1) in each of the six sectors in the complex k-plane, as shown in Figure 2.8.

Theorem 3.3.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $T_{\mathrm{l}}(k)$ and $T_{\mathrm{r}}(k)$ be the respective left and right transmission coefficients for (1.1) appearing in (2.35) and (2.39). Similarly, let $\overline{T}_{\mathrm{l}}(k)$ and $\overline{T}_{\mathrm{r}}(k)$ be the respective left* *and right transmission coefficients for* (2.49) *appearing in* (3.1) *and* (3.5)*. Then, we have the following:*

(a) *The adjoint right transmission coefficient* $\overline{T}_{\rm r}(k)$ *is related to the left transmission coefficient* $T_{\rm l}(k)$ *as*

$$\overline{T}_{\rm r}(-k^*)^* = T_{\rm l}(k), \qquad k \in \overline{\Omega}_1. \tag{3.19}$$

(b) *The adjoint left transmission coefficient* $\overline{T}_{\rm l}(k)$ *is related to the right transmission coefficient* $T_{\rm r}(k)$ *as*

$$\overline{T}_{\rm l}(-k^*)^* = T_{\rm r}(k), \qquad k \in \overline{\Omega}_3. \tag{3.20}$$

Proof. The proof is obtained by evaluating the 3-Wronskians of the solutions to (1.1) listed in the six open sectors shown in Figure 2.8 and by using the identities obtained by exploiting the fact that each of those 3-Wronskians has the same value as $x \to +\infty$ and $x \to -\infty$. ■

Having established the relationships between the transmission and adjoint transmission coefficients in Theorem 3.3.1, in Figure 3.1 we show those relationships everywhere in the complex k-plane. We recall that the k-domain of $T_{\rm l}(zk)$ is obtained by rotating the k-domain of $T_{\rm l}(k)$ clockwise by $2\pi/3$ in the complex k-plane. Similarly, the k-domain of $T_{\rm l}(z^2k)$ is obtained by rotating the k-domain of $T_{\rm l}(k)$ counterclockwise by $2\pi/3$ in $\mathbb{C}$. The k-domains of $T_{\rm r}(zk)$ and $T_{\rm r}(z^2k)$ are obtained from the k-domain of $T_{\rm r}(k)$ in a similar manner.

In the next theorem, we present the relationships between the scattering coefficients for (1.1) and the adjoint scattering coefficients for (2.49). This is done by evaluating the Wronkskians of three relevant solutions to (1.1) on the boundaries $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, where the relevant solutions are shown in Figure 2.8.

Theorem 3.3.2. *Assume that the potentials* $Q(x)$ *and* $P(x)$ *in* (1.1) *belong to the Schwartz class* $\mathcal{S}(\mathbb{R})$*. Let* $T_{\rm l}(k)$*,* $L(k)$*,* $M(k)$*, and* $T_{\rm r}(k)$*,* $R(k)$*,* $N(k)$ *be the respective left and right scattering coefficients for* (1.1) *appearing in* (2.35) *and* (2.39)*. Similarly, let* $\overline{T}_{\rm l}(k)$*,* $\overline{L}(k)$*,* $\overline{M}(k)$*, and* $\overline{T}_{\rm r}(k)$*,* $\overline{R}(k)$*,* $\overline{N}(k)$ *be the respective left and right adjoint scattering coefficients for* (2.49) *appearing in* (3.1) *and* (3.5)*. Then, we have the following:*

(a) *The adjoint right primary reflection coefficient* $\overline{R}(k)$ *is related to the left primary reflection coefficient* $L(k)$ *as*

$$\overline{R}(-z^2k^*)^* \left[\overline{T}_{\rm r}(-z^2k^*)^*\right]^{-1} = z^2 L(k)\, T_{\rm l}(k)^{-1}, \qquad k \in \mathcal{L}_1, \tag{3.21}$$

where we recall that the boundary $\mathcal{L}_1$ *can be parametrized as* $k = zs$ *for* $s \geq 0$ *and* z *is the special constant in* (2.1).

(b) *The adjoint right secondary reflection coefficient* $\overline{N}(k)$ *is related to the left secondary reflection coefficient* $M(k)$ *as*

$$\overline{N}(-zk^*)^* \left[\overline{T}_{\rm r}(-zk^*)^*\right]^{-1} = z\, M(k)\, T_{\rm l}(k)^{-1}, \qquad k \in \mathcal{L}_2, \tag{3.22}$$

where we recall that the boundary $\mathcal{L}_2$ *can be parametrized as* $k = z^2 s$ *for* $s \geq 0$.

(c) *The adjoint left primary reflection coefficient* $\overline{L}(k)$ *is related to the right primary reflection coefficient* $R(k)$ *as*

$$\overline{L}(-z^2k^*)^* \left[\overline{T}_{\rm l}(-z^2k^*)^*\right]^{-1} = z^2\, R(k)\, T_{\rm r}(k)^{-1}, \qquad k \in \mathcal{L}_3, \tag{3.23}$$

where we recall that the boundary $\mathcal{L}_3$ *can be parametrized as* $k = -zs$ *for* $s \geq 0$.

(d) *The adjoint left secondary reflection coefficient* $\overline{M}(k)$ *is related to the right secondary reflection coefficient* $N(k)$ *as*

$$\overline{M}(-zk^*)^* \left[\overline{T}_{\rm l}(-zk^*)^*\right]^{-1} = z\, N(k)\, T_{\rm r}(k)^{-1}, \qquad k \in \mathcal{L}_4, \tag{3.24}$$

where we recall that the boundary $\mathcal{L}_4$ *can be parametrized as* $k = -z^2 s$ *for* $s \geq 0$.

Proof. As seen from Figure 2.8, on each directed half line $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ in the complex k-plane, we have five distinct solutions to (1.1). Thus, on each of those four half lines we can construct ten 3-Wronskians from those five solutions to (1.1). By exploiting the fact that each of those 3-Wronskians yields the same value as $x \to +\infty$ and $x \to -\infty$, we obtain the identities listed in (3.21)–(3.24). We remark that in the evaluations of those 3-Wronskians as $x \to \pm\infty$, we use the spacial asymptotics listed in (2.21), (2.29), and Theorems 2.5.1, 2.5.2, 3.2.1, and 3.2.2. ■

In Theorems 3.2.1 and 3.2.2, we have presented the respective spacial asymptotics of the solutions $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ to (1.1). Those asymptotics have been expressed in terms of the adjoint scattering coefficients for (2.49). Using the results in Theorems 3.3.1 and 3.3.2, we can express the same asymptotics in terms of the scattering coefficients for (1.1). This is done in the next corollary.

Corollary 3.3.2.1. *Assume that $Q(x)$ and $P(x)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $h^{\text{down}}(k,x)$ be the solution to (1.1) given in (2.57), and let $h^{\text{up}}(k,x)$ be the solution to (1.1) given in (2.58). We have the following:*

(a) *For $k \in \mathcal{L}_2$, we have the spacial asymptotics of $h^{\text{down}}(k,x)$ given by*

$$h^{\text{down}}(k,x) = \begin{cases} z(1-z)k\, e^{kx}\, T_{\text{l}}(z^2k)^{-1} \\ \quad -z(1-z)k\, e^{z^2kx}\, M(k)\, T_{\text{l}}(k)^{-1} + o(1), \\ z(1-z)k\, e^{kx}\, T_{\text{r}}(zk)^{-1} + o(1), \end{cases} \tag{3.25}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$.

(b) *For $k \in \mathcal{L}_3$, we have the spacial asymptotics of $h^{\text{down}}(k,x)$ given by*

$$h^{\text{down}}(k,x) = \begin{cases} z(1-z)k\, e^{kx}\, T_{\text{l}}(z^2k)^{-1} + o(1), \\ z(1-z)k\, e^{kx}\, T_{\text{r}}(zk)^{-1} \\ \quad -z(1-z)k\, e^{zkx}\, R(k)\, T_{\text{r}}(k)^{-1} + o(1), \end{cases} \tag{3.26}$$

where the top line on the right-hand side refers to the asymptotics when $x \to +\infty$ and the bottom line refers to the asymptotics when $x \to -\infty$.

(c) *For $k \in \Omega_2$, the spacial asymptotics of $h^{\text{down}}(k,x)$ are given by*

$$h^{\text{down}}(k,x) = \begin{cases} z(1-z)k\, e^{kx}\, T_{\text{l}}(z^2k)^{-1} + o(1), & x \to +\infty, \\ z(1-z)k\, e^{kx}\, T_{\text{r}}(zk)^{-1} + o(1), & x \to -\infty. \end{cases} \tag{3.27}$$

(d) *For* $k\in\mathcal{L}_1$*, the spacial asymptotics of* $h^{\text{up}}(k,x)$ *is given by*

$$h^{\text{up}}(k,x)=\begin{cases}-z(1-z)k\,e^{kx}\,T_{\text{l}}(zk)^{-1}\\ \quad+z(1-z)k\,e^{zkx}\,L(k)\,T_{\text{l}}(k)^{-1}+o(1),\\ -z(1-z)k\,e^{kx}\,T_{\text{r}}(z^2k)^{-1}+o(1),\end{cases}\tag{3.28}$$

where the top line on the right-hand side refers to the asymptotics when $x\to+\infty$ *and the bottom line refers to the asymptotics when* $x\to-\infty$*.*

(e) *For* $k\in\mathcal{L}_4$*, the spacial asymptotics of* $h^{\text{up}}(k,x)$ *is given by*

$$h^{\text{up}}(k,x)=\begin{cases}-z(1-z)k\,e^{kx}\,T_{\text{l}}(zk)^{-1}+o(1),\\ -z(1-z)k\,e^{kx}\,T_{\text{r}}(z^2k)^{-1}\\ \quad-z(1-z)k\,e^{z^2kx}\,N(k)\,T_{\text{r}}(k)^{-1}+o(1),\end{cases}\tag{3.29}$$

where the top line on the right-hand side refers to the asymptotics when $x\to+\infty$ *and the bottom line refers to the asymptotics when* $x\to-\infty$*.*

(f) *For* $k\in\Omega_4$*, the spacial asymptotics of* $h^{\text{up}}(k,x)$ *are given by*

$$h^{\text{up}}(k,x)=\begin{cases}-z(1-z)k\,e^{kx}\,T_{\text{l}}(zk)^{-1}+o(1), & x\to+\infty,\\ -z(1-z)k\,e^{kx}\,T_{\text{r}}(z^2k)^{-1}+o(1), & x\to-\infty.\end{cases}\tag{3.30}$$

As a direct consequence of Theorem 3.3.2, in the next corollary we express the four adjoint reflection coefficients for (2.49) in terms of the scattering coefficients for (1.1).

Corollary 3.3.2.2. *Assume that the potentials* $Q(x)$ *and* $P(x)$ *in* (1.1) *belong to the Schwartz class* $\mathcal{S}(\mathbb{R})$*. Let* $T_{\text{l}}(k)$*,* $L(k)$*,* $M(k)$*, and* $T_{\text{r}}(k)$*,* $R(k)$*,* $N(k)$ *be the respective left and right scattering coefficients for* (1.1) *appearing in* (2.35) *and* (2.39)*. Let* $\overline{L}(k)$*,* $\overline{M}(k)$*,* $\overline{R}(k)$*,* $\overline{N}(k)$ *be the respective adjoint reflection coefficients for* (2.49) *appearing in* (3.1) *and* (3.5)*. Then, we have the following:*

(a) *The adjoint right primary reflection coefficient* $\overline{R}(k)$ *is expressed in terms of the left scattering coefficients for* (1.1) *as*

$$\overline{R}(-z^2k^*)^*=\frac{z^2L(k)\,T_{\text{l}}(k)^{-1}}{T_{\text{l}}(zk)^{-1}},\qquad k\in\mathcal{L}_1.\tag{3.31}$$

(b) *The adjoint right secondary reflection coefficient* $\overline{N}(k)$ *is expressed in terms of the left scattering coefficients for* (1.1) *as*

$$\overline{N}(-zk^*)^* = \frac{z\, M(k)\, T_{\rm l}(k)^{-1}}{T_{\rm l}(z^2k)^{-1}}, \qquad k \in \mathcal{L}_2. \tag{3.32}$$

(c) *The adjoint left primary reflection coefficient* $\overline{L}(k)$ *is expressed in terms of the right scattering coefficients for* (1.1) *as*

$$\overline{L}(-z^2k^*)^* = \frac{z^2\, R(k)\, T_{\rm r}(k)^{-1}}{T_{\rm r}(zk)^{-1}}, \qquad k \in \mathcal{L}_3. \tag{3.33}$$

(d) *The adjoint left secondary reflection coefficient* $\overline{M}(k)$ *is expressed in terms of the right scattering coefficients for* (1.1) *as*

$$\overline{M}(-zk^*)^* = \frac{z\, N(k)\, T_{\rm r}(k)^{-1}}{T_{\rm r}(z^2k)^{-1}}, \qquad k \in \mathcal{L}_4. \tag{3.34}$$

In the next theorem, we relate the four reflection coefficients $L(k)$, $M(k)$, $R(k)$, $N(k)$ to the two transmission coefficients $T_{\rm l}(k)$ and $T_{\rm r}(k)$ for (1.1).

Theorem 3.3.3. *Assume that the potentials* $Q(x)$ *and* $P(x)$ *in* (1.1) *belong to the Schwartz class* $\mathcal{S}(\mathbb{R})$*. Let* $T_{\rm l}(k)$*,* $L(k)$*,* $M(k)$*, and* $T_{\rm r}(k)$*,* $R(k)$*,* $N(k)$ *be the respective left and right scattering coefficients for* (1.1) *appearing in* (2.35) *and* (2.39)*. Then, we have the following:*

(a) *The left reflection coefficients* $L(k)$ *and* $M(k)$ *are related to the transmission coefficients* $T_{\rm l}(k)$ *and* $T_{\rm r}(k)$ *as*

$$T_{\rm r}(z^2k)^{-1} = T_{\rm l}(k)^{-1}\, T_{\rm l}(zk)^{-1}\, [1 - L(k)\, M(zk)], \qquad k \in \mathcal{L}_1. \tag{3.35}$$

(b) *The right reflection coefficients* $R(k)$ *and* $N(k)$ *are related to the transmission coefficients* $T_{\rm l}(k)$ *and* $T_{\rm r}(k)$ *as*

$$T_{\rm l}(z^2k)^{-1} = T_{\rm r}(k)^{-1}\, T_{\rm r}(zk)^{-1}\, [1 - R(k)\, N(zk)]\,, \qquad k \in \mathcal{L}_3. \tag{3.36}$$

Proof. The relationships listed in (3.35) and (3.36) are obtained by evaluating the 3-Wronskian of $f(k,x)$, $f(zk,x)$, $g(z^2k,x)$ on the directed half line $\mathcal{L}_1$ and the 3-Wronskian of $g(k,x)$, $g(zk,x)$, $f(z^2k,x)$ on $\mathcal{L}_3$ and by using the fact that each of those 3-Wronskians yields the same value as $x \to +\infty$ and $x \to +\infty$. ■

3.4 The large k-asymptotics for (1.1)

The spacial asymptotics for the four fundamental solutions $f(k,x)$, $g(k,x)$, $h^{\text{down}}(k,x)$, $h^{\text{up}}(k,x)$ to (1.1) have been established in the previous section. In this section we establish the large k-asymptotics of those fundamental solutions. We also present the large k-asymptotics for the scattering coefficients for (1.1).

We recall that the left and right Jost solutions $f(k,x)$ and $g(k,x)$ are related to the auxiliary functions $u(k,x)$ and $v(k,x)$ appearing in (2.20) and (2.28), respectively. It is more convenient to obtain first the large k-asymptotics of $u(k,x)$ and $v(k,x)$ and then use (2.20) and (2.28) to recover the large k-asymptotics of $f(k,x)$ and $g(k,x)$, respectively. This is because the large k-asymptotics of $u(k,x)$ and $v(k,x)$ can readily be evaluated from the respective integral equations (2.25) and (2.33).

In the next theorem, we present the large k-asymptotics of the four fundamental solutions to (1.1).

Theorem 3.4.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $u(k,x)$ and $v(k,x)$ be the auxiliary quantities related to the Jost solutions $f(k,x)$ and $g(k,x)$ to (1.1) as in (2.20) and (2.28), respectively. Let $h^{\text{down}}(k,x)$ and $h^{\text{up}}(k,x)$ be the solutions to (1.1) appearing in (2.57) and (2.58), respectively. Then for each fixed $x \in \mathbb{R}$, we have the following large k-asymptotics:*

(a) *As $k \to \infty$ in $\overline{\Omega}_1$, the quantity $u(k,x)$ has the asymptotic behavior*

$$u(k,x) = 1 + \frac{u_1(x)}{k} + \frac{u_2(x)}{k^2} + O\left(\frac{1}{k^3}\right), \tag{3.37}$$

where we have defined

$$u_1(x) := \frac{1}{3}\int_x^\infty dy\, Q(y), \tag{3.38}$$

$$u_2(x) := -\frac{1}{3}\int_x^\infty dy\, \left[Q'(y) - P(y)\right] + \frac{1}{18}\left[\int_x^\infty dy\, Q(y)\right]^2, \tag{3.39}$$

and we recall that $\overline{\Omega}_1$ is the closed sector described in (2.7) and shown in Figure 2.1.

(b) *Consequently, as $k \to \infty$ in $\overline{\Omega}_1$, the left Jost solution $f(k,x)$ has the asymptotic behavior*

$$f(k,x) = e^{kx}\left[1 + \frac{u_1(x)}{k} + \frac{u_2(x)}{k^2} + O\left(\frac{1}{k^3}\right)\right]. \tag{3.40}$$

(c) *As $k \to \infty$ in $\overline{\Omega}_3$, the quantity $v(k,x)$ has the asymptotic behavior*

$$v(k,x) = 1 + \frac{v_1(x)}{k} + \frac{v_2(x)}{k^2} + O\left(\frac{1}{k^3}\right), \tag{3.41}$$

where we have defined

$$v_1(x) := -\frac{1}{3}\int_{-\infty}^{x} dy\, Q(y), \tag{3.42}$$

$$v_2(x) := \frac{1}{3}\int_{-\infty}^{x} dy\, \left[Q'(y) - P(y)\right] + \frac{1}{18}\left[\int_{-\infty}^{x} dy\, Q(y)\right]^2, \tag{3.43}$$

and we recall that $\overline{\Omega}_3$ is the closed sector described in (2.9) *and shown in Figure 2.1.*

(d) *Consequently, as $k \to \infty$ in $\overline{\Omega}_3$, the right Jost solution $g(k,x)$ has the asymptotic behavior*

$$g(k,x) = e^{kx}\left[1 + \frac{v_1(x)}{k} + \frac{v_2(x)}{k^2} + O\left(\frac{1}{k^3}\right)\right]. \tag{3.44}$$

(e) *As $k \to \infty$ in $\overline{\Omega}_2$, the solution $h^{\rm down}(k,x)$ has the asymptotic behavior*

$$\frac{h^{\rm down}(k,x)}{z(1-z)k} = e^{kx}\left[1 + \frac{w_1^{\rm down}(x)}{k} + \frac{w_2^{\rm down}(x)}{k^2} + O\left(\frac{1}{k^3}\right)\right], \tag{3.45}$$

where we have defined

$$w_1^{\rm down}(x) := \frac{1}{3z^2}\int_{-\infty}^{x} dy\, Q(y) - \frac{1}{3z}\int_{x}^{\infty} dy\, Q(y), \tag{3.46}$$

$$\begin{aligned} w_2^{\rm down}(x) :=& \frac{1}{3}Q(x) + \frac{1}{3z}\int_{-\infty}^{x} dy\, P(y) - \frac{1}{3z^2}\int_{x}^{\infty} dy\, P(y) \\ &+ \frac{1}{18z}\left(\int_{-\infty}^{x} dy\, Q(y)\right)^2 + \frac{1}{18z^2}\left(\int_{x}^{\infty} dy\, Q(y)\right)^2 \\ &- \frac{1}{9}\left(\int_{x}^{\infty} dy\, Q(y)\right)\left(\int_{-\infty}^{x} dy\, Q(y)\right), \end{aligned} \tag{3.47}$$

and we recall that $\overline{\Omega}_2$ is the closed sector shown in Figure 2.1.

(f) *As $k \to \infty$ in $\overline{\Omega}_4$, the solution $h^{\rm up}(k,x)$ has the asymptotic behavior*

$$\frac{h^{\rm up}(k,x)}{-z(1-z)k} = e^{kx}\left[1 + \frac{w_1^{\rm up}(x)}{k} + \frac{w_2^{\rm up}(x)}{k^2} + O\left(\frac{1}{k^3}\right)\right], \tag{3.48}$$

where we have defined

$$w_1^{up}(x) := \frac{1}{3z}\int_{-\infty}^{x} dy\, Q(y) - \frac{1}{3z^2}\int_{x}^{\infty} dy\, Q(y), \tag{3.49}$$

$$\begin{aligned} w_2^{up}(x) :=& \frac{1}{3}Q(x) + \frac{1}{3z^2}\int_{-\infty}^{x} dy\, P(y) - \frac{1}{3z}\int_{x}^{\infty} dy\, P(y) \\ &+ \frac{1}{18z^2}\left(\int_{-\infty}^{x} dy\, Q(y)\right)^2 + \frac{1}{18z}\left(\int_{x}^{\infty} dy\, Q(y)\right)^2 \\ &- \frac{1}{9}\left(\int_{x}^{\infty} dy\, Q(y)\right)\left(\int_{-\infty}^{x} dy\, Q(y)\right), \end{aligned} \tag{3.50}$$

and we recall that $\overline{\Omega}_2$ *is the closed sector shown in Figure 2.1.*

Proof. We obtain the large k-asymptotics given in (3.37) directly from the integral representation of $u(k,x)$ in (2.25). Hence, the proof of (a) is complete. We remark that (b) directly follows from (a) by using the connection between $f(k,x)$ and $u(k,x)$ given in (2.22). The proof of (c) is obtained by using the large k-asymptotics of $v(k,x)$ with the help of its integral representation in (2.33). The proof of (d) directly follows from (c). We obtain the large k-asymptotics in (e) by using the definition of $h^{\text{down}}(k,x)$ given in (2.57) expressed in terms of the adjoint Jost solutions $\overline{f}(k,x)$ and $\overline{g}(k,x)$ and by using the large k-asymptotics of those adjoint Jost solutions. The large k-asymptotics of the adjoint Jost solutions are similar to the large k-asymptotics of the Jost solutions expressed in (b) and (d) and they can be expressed in terms of the potentials $Q(x)$ and $P(x)$ after using (2.50). In a similar manner, we obtain the large k-asymptotics in (f) by using the definition of $h^{\text{up}}(k,x)$ given in (2.58) and using the large k-asymptotics of the adjoint Jost solutions. ■

In the next theorem, we present the large k-asymptotics of the scattering coefficients for (1.1).

Theorem 3.4.2. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $T_{\rm l}(k)$, $L(k)$, $M(k)$, and $T_{\rm r}(k)$, $R(k)$, $N(k)$ be the left and right scattering coefficients for (1.1) appearing in (2.35) and (2.39), respectively. We have the following large k-asymptotics:*

(a) *As* $k \to \infty$ *in* $\overline{\Omega}_1$*, the reciprocal of the left transmission coefficient* $T_{\rm l}(k)$ *has the asymptotic behavior*

$$T_{\rm l}(k)^{-1} = 1 + \frac{u_1(-\infty)}{k} + \frac{u_2(-\infty)}{k^2} + O\left(\frac{1}{k^3}\right), \tag{3.51}$$

where we have defined

$$u_1(-\infty) := \frac{1}{3}\int_{-\infty}^{\infty} dy\, Q(y), \tag{3.52}$$

$$u_2(-\infty) := -\frac{1}{3}\int_{-\infty}^{\infty} dy\; [Q'(y) - P(y)] + \frac{1}{18}\left(\int_{-\infty}^{\infty} dy\, Q(y)\right)^2. \tag{3.53}$$

We remark that $u_1(-\infty)$ *and* $u_2(-\infty)$ *are obtained from* $u_1(x)$ *and* $u_2(x)$ *appearing in* (3.38) *and* (3.39)*, respectively, by letting* $x \to -\infty$ *there.*

(b) *Consequently, as* $k \to \infty$ *in* $\overline{\Omega}_1$*, the left transmission coefficient* $T_{\rm l}(k)$ *has the asymptotic behavior*

$$T_{\rm l}(k) = 1 - \frac{u_1(-\infty)}{k} + O\left(\frac{1}{k^2}\right), \tag{3.54}$$

(c) *As* $k \to \infty$ *in* $\overline{\Omega}_3$*, the reciprocal of the right transmission coefficient* $T_{\rm r}(k)$ *has the asymptotic behavior*

$$T_{\rm r}(k)^{-1} = 1 + \frac{v_1(+\infty)}{k} + \frac{v_2(+\infty)}{k^2} + O\left(\frac{1}{k^3}\right), \tag{3.55}$$

where we have defined

$$v_1(+\infty) := -\frac{1}{3}\int_{-\infty}^{\infty} dy\, Q(y), \tag{3.56}$$

$$v_2(+\infty) := \frac{1}{3}\int_{-\infty}^{\infty} dy\; [Q'(y) - P(y)] + \frac{1}{18}\left(\int_{-\infty}^{\infty} dy\, Q(y)\right)^2. \tag{3.57}$$

We remark that $v_1(+\infty)$ *and* $v_2(+\infty)$ *are obtained from* $v_1(x)$ *and* $v_2(x)$ *appearing in* (3.42) *and* (3.43)*, respectively, by letting* $x \to +\infty$ *there.*

(d) *Consequently, as* $k \to \infty$ *in* $\overline{\Omega}_3$*, the right transmission coefficient* $T_{\rm r}(k)$ *has the asymptotic behavior*

$$T_{\rm r}(k) = 1 - \frac{v_1(+\infty)}{k} + O\left(\frac{1}{k^2}\right), \tag{3.58}$$

(e) *As $k \to \infty$ on $\mathcal{L}_1$, the left primary reflection coefficient $L(k)$ has the asymptotics*

$$L(k) = \frac{z^2}{3k} \int_{-\infty}^{\infty} dy\, e^{\sqrt{3}iz^2ky}\, Q(y) + O\left(\frac{1}{k^2}\right), \tag{3.59}$$

where we recall that $\mathcal{L}_1$ is the directed half line in complex k-plane shown in Figure 2.1 and has the parametrization $k = zs$ for $s \geq 0$.

(f) *As $k \to \infty$ on $\mathcal{L}_2$, the left secondary reflection coefficient $M(k)$ has the asymptotics*

$$M(k) = \frac{z}{3k} \int_{-\infty}^{\infty} dy\, e^{-\sqrt{3}izky}\, Q(y) + O\left(\frac{1}{k^2}\right), \tag{3.60}$$

where we recall that $\mathcal{L}_2$ is the directed half line in the complex k-plane shown in Figure 2.1 and has the parametrization $k = z^2 s$ for $s \geq 0$.

(g) *As $k \to \infty$ on $\mathcal{L}_3$, the right primary reflection coefficient $R(k)$ has the asymptotics*

$$R(k) = -\frac{z^2}{3k} \int_{-\infty}^{\infty} dy\, e^{\sqrt{3}iz^2ky}\, Q(y) + O\left(\frac{1}{k^2}\right), \tag{3.61}$$

where we recall that $\mathcal{L}_3$ is the directed half line in the complex k-plane shown in Figure 2.1 and has the parametrization $k = -zs$ for $s \geq 0$.

(h) *As $k \to \infty$ on $\mathcal{L}_4$, the left secondary reflection coefficient $N(k)$ has the asymptotics*

$$N(k) = -\frac{z}{3k} \int_{-\infty}^{\infty} dy\, e^{-\sqrt{3}izky}\, Q(y) + O\left(\frac{1}{k^2}\right), \tag{3.62}$$

where we recall that $\mathcal{L}_4$ is the directed half line in the complex k-plane shown in Figure 2.1 and has the parametrization $k = -z^2 s$ for $s \geq 0$.

Proof. We obtain the large k-asymptotics in (a) by using the integral representation of $T_{\mathrm{l}}(k)^{-1}$ given in (2.36). The result in (b) is a direct consequence of (a). The large k-asymptotics in (c) is obtained by using the integral representation of $T_{\mathrm{r}}(k)^{-1}$ given in (2.40). The result in (d) follows from (c). We obtain (3.59) in (e) by using (2.37) and (3.54). We establish (3.60) in (f) with the help of (2.38) and (3.54). We obtain (3.61) in (g) by using (2.41) and (3.58). Finally, we get (3.62) in (h) with the help of (2.42) and (3.58). ■

3.5 The small k-asymptotics of the scattering coefficients for (1.1)

In the previous section, we have presented the large k-asymptotics of the scattering coefficients for (1.1). In this section we determine the small k-asymptotics of those scattering coefficients. In the next theorem we present the relevant small k-asymptotics related to the scattering coefficients for (1.1).

Theorem 3.5.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $T_{\rm l}(k)$, $L(k)$, $M(k)$, and $T_{\rm r}(k)$, $R(k)$, $N(k)$ be the left and right scattering coefficients for (1.1) appearing in (2.35) and (2.39), respectively. We have the following small k-asymptotics:*

(a) *As $k \to 0$ in $\overline{\Omega}_1$, the reciprocal of the left transmission coefficient $T_{\rm l}(k)$ has the asymptotics*

$$T_{\rm l}(k)^{-1} = \frac{\alpha_{-2}}{k^2} + \frac{\alpha_{-1}}{k} + O(1), \tag{3.63}$$

where we have defined

$$\alpha_{-2} := -\frac{1}{3}\int_{-\infty}^{\infty} dy\,[Q'(y) - P(y)]\,u(0,y), \tag{3.64}$$

$$\alpha_{-1} := \frac{1}{3}\int_{-\infty}^{\infty} dy\Big(Q(y)\,u(0,y) - [Q'(y) - P(y)]\,\dot{u}(0,y)\Big), \tag{3.65}$$

with $\dot{u}(0,y)$ denoting the k-derivative of $u(k,y)$ evaluated at $k=0$.

(b) *Similarly, as $k \to 0$ in $\overline{\Omega}_3$, the reciprocal of the right transmission coefficient $T_{\rm r}(k)$ has the asymptotics*

$$T_{\rm r}(k)^{-1} = \frac{\beta_{-2}}{k^2} + \frac{\beta_{-1}}{k} + O(1), \qquad k \to 0, \tag{3.66}$$

where we have defined

$$\beta_{-2} := \frac{1}{3}\int_{-\infty}^{\infty} dy\,[Q'(y) - P(y)]\,v(0,y), \tag{3.67}$$

$$\beta_{-1} := -\frac{1}{3}\int_{-\infty}^{\infty} dy\Big(Q(y)\,v(0,y) - [Q'(y) - P(y)]\,\dot{v}(0,y)\Big), \tag{3.68}$$

with $\dot{v}(0,y)$ denoting the k-derivative of $v(k,y)$ evaluated at $k=0$.

(c) *As $k \to 0$ on $\mathcal{L}_1$, we have*

$$L(k)\,T_{\rm l}(k)^{-1} = \frac{z\,\alpha_{-2}}{k^2} + \frac{\gamma_{-1}}{k} + O(1), \tag{3.69}$$

where we have defined

$$\begin{aligned}\gamma_{-1} :=& \frac{1}{3}\int_{-\infty}^{\infty} dy \left(z^2\, Q(y) - \sqrt{3}iy[Q'(y) - P(y)]\right) u(0,y) \\ &- \frac{z}{3}\int_{-\infty}^{\infty} dy\,[Q'(y) - P(y)]\,\dot{u}(0,y).\end{aligned} \tag{3.70}$$

(d) *As $k \to 0$ on $\mathcal{L}_2$, we have*

$$M(k)\,T_{\rm l}(k)^{-1} = \frac{z^2\,\alpha_{-2}}{k^2} + \frac{\delta_{-1}}{k} + O(1), \tag{3.71}$$

where we have defined

$$\begin{aligned}\delta_{-1} :=& \frac{1}{3}\int_{-\infty}^{\infty} dy \left(z\, Q(y) + \sqrt{3}iy[Q'(y) - P(y)]\right) u(0,y) \\ &- \frac{z^2}{3}\int_{-\infty}^{\infty} dy\,[Q'(y) - P(y)]\,\dot{u}(0,y).\end{aligned} \tag{3.72}$$

(e) *As $k \to 0$ on $\mathcal{L}_3$, we have*

$$R(k)\,T_{\rm r}(k)^{-1} = \frac{z\,\beta_{-2}}{k^2} + \frac{\omega_{-1}}{k} + O(1), \tag{3.73}$$

where we have defined

$$\begin{aligned}\omega_{-1} :=& -\frac{1}{3}\int_{-\infty}^{\infty} dy \left(z^2\, Q(y) - \sqrt{3}iy[Q'(y) - P(y)]\right) v(0,y) \\ &+ \frac{z}{3}\int_{-\infty}^{\infty} dy\,[Q'(y) - P(y)]\,\dot{v}(0,y).\end{aligned} \tag{3.74}$$

(f) *As $k \to 0$ on $\mathcal{L}_4$, we have*

$$N(k)\,T_{\rm r}(k)^{-1} = \frac{z^2\,\beta_{-2}}{k^2} + \frac{\eta_{-1}}{k} + O(1), \tag{3.75}$$

where we have defined

$$\begin{aligned}\eta_{-1} :=& -\frac{1}{3}\int_{-\infty}^{\infty} dy \left(z\, Q(y) + \sqrt{3}iy[Q'(y) - P(y)]\right) v(0,y) \\ &+ \frac{z^2}{3}\int_{-\infty}^{\infty} dy\,[Q'(y) - P(y)]\,\dot{v}(0,y).\end{aligned} \tag{3.76}$$

Proof. The proof of (a) follows from the representation of $T_{\rm l}(k)^{-1}$ given in (2.36) and by using the small k-expansion of $u(k,x)$ given by

$$u(k,x) = u(0,x) + k\,\dot{u}(0,x) + O(k^2), \qquad k \to 0 \text{ in } \overline{\Omega}_1, \tag{3.77}$$

where we recall that $\dot{u}(0,x)$ denotes the k-derivative of $u(k,x)$ evaluated at $k=0$. The proof of (b) is obtained in a similar manner by using the representation of $T_{\rm r}(k)^{-1}$ given in (2.40) and by using the small k-expansion of $v(k,x)$ given by

$$v(k,x) = v(0,x) + k\,\dot{v}(0,x) + O(k^2), \qquad k \to 0 \text{ in } \overline{\Omega}_3. \tag{3.78}$$

The results in (c), (d), (e), (f) are obtained by using the representation of $L(k)\,T_{\rm l}(k)^{-1}$ in (2.37), $M(k)\,T_{\rm l}(k)^{-1}$ in (2.38), $R(k)\,T_{\rm r}(k)^{-1}$ in (2.41), $N(k)\,T_{\rm r}(k)^{-1}$ in (2.42), respectively, and with the help of the expansions in (3.77) and (3.78). ∎

By using the small k-asymptotics of the relevant quantities given in Theorem 3.5.1, in the following corollary we present the small k-asymptotics for the scattering coefficients for (1.1).

Corollary 3.5.1.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $T_{\rm l}(k)$, $L(k)$, $M(k)$, and $T_{\rm r}(k)$, $R(k)$, $N(k)$ be the left and right scattering coefficients for (1.1) appearing in (2.35) and (2.39), respectively. We have the following small k-asymptotics:*

(a) *If the constant α_{-2} defined in (3.64) is nonzero, then the left transmission coefficient $T_{\rm l}(k)$ is continuous at $k=0$ and it has the small k-asymptotics given by*

$$T_{\rm l}(k) = \frac{k^2}{\alpha_{-2}} + O(k^3), \qquad k \to 0 \text{ in } \overline{\Omega}_1. \tag{3.79}$$

(b) *If the constant α_{-2} defined in (3.64) is zero and the constant α_{-1} defined in (3.65) is nonzero, then the left transmission coefficient $T_{\rm l}(k)$ is continuous at $k=0$ and it has the small k-asymptotics given by*

$$T_{\rm l}(k) = \frac{k}{\alpha_{-1}} + O(k^2), \qquad k \to 0 \text{ in } \overline{\Omega}_1. \tag{3.80}$$

(c) *If the constant* α_{-2} *defined in* (3.64) *and the constant* a_{-1} *defined in* (3.65) *are both zero, then the left transmission coefficient* $T_{\rm l}(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$T_{\rm l}(k) = O(1), \qquad k \to 0 \text{ in } \overline{\Omega}_1. \tag{3.81}$$

(d) *If the constant* β_{-2} *defined in* (3.67) *is nonzero, then the right transmission coefficient* $T_{\rm r}(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$T_{\rm r}(k) = \frac{k^2}{\beta_{-2}} + O(k^3), \qquad k \to 0 \text{ in } \overline{\Omega}_3. \tag{3.82}$$

(e) *If the constant* β_{-2} *defined in* (3.67) *is zero and the constant* β_{-1} *defined in* (3.68) *is nonzero, then the right transmission coefficient* $T_{\rm r}(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$T_{\rm r}(k) = \frac{k}{\beta_{-1}} + O(k^2), \qquad k \to 0 \text{ in } \overline{\Omega}_3. \tag{3.83}$$

(f) *If the constant* β_{-2} *defined in* (3.67) *and the constant* β_{-1} *defined in* (3.68) *are both zero, then the right transmission coefficient* $T_{\rm r}(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$T_{\rm r}(k) = O(1), \qquad k \to 0 \text{ in } \overline{\Omega}_3. \tag{3.84}$$

(g) *If the constant* α_{-2} *defined in* (3.64) *is nonzero, then the left primary reflection coefficient* $L(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$L(k) = z + O(k), \qquad k \to 0 \text{ in } \mathcal{L}_1. \tag{3.85}$$

(h) *If the constant* α_{-2} *defined in* (3.64) *is zero and the constant* α_{-1} *defined in* (3.65) *is nonzero, then the left primary reflection coefficient* $L(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$L(k) = \frac{\gamma_{-1}}{\alpha_{-1}} + O(k), \qquad k \to 0 \text{ in } \mathcal{L}_1, \tag{3.86}$$

where γ_{-1} *is the constant defined in* (3.70).

(i) *If the constant* α_{-2} *defined in* (3.64) *and the constant* α_{-1} *defined in* (3.65) *are both zero, then the left primary reflection coefficient* $L(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$L(k) = O(1), \qquad k \to 0 \text{ in } \mathcal{L}_1. \tag{3.87}$$

(j) *If the constant* α_{-2} *defined in* (3.64) *is nonzero, then the left secondary reflection coefficient* $M(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$M(k) = z^2 + O(k), \qquad k \to 0 \text{ in } \mathcal{L}_2. \tag{3.88}$$

(k) *If the constant* α_{-2} *defined in* (3.64) *is zero and the constant* α_{-1} *defined in* (3.65) *is nonzero, then the left secondary reflection coefficient* $M(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$M(k) = \frac{\delta_{-1}}{\alpha_{-1}} + O(k), \qquad k \to 0 \text{ in } \mathcal{L}_2, \tag{3.89}$$

where δ_{-1} *is the constant defined in* (3.72)*.*

(l) *If the constant* α_{-2} *defined in* (3.64) *and the constant* α_{-1} *defined in* (3.65) *are both zero, then the left secondary reflection coefficient* $M(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$M(k) = O(1), \qquad k \to 0 \text{ in } \mathcal{L}_2. \tag{3.90}$$

(m) *If the constant* β_{-2} *defined in* (3.67) *is nonzero, then the right primary reflection coefficient* $R(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$R(k) = z + O(k), \qquad k \to 0 \text{ in } \mathcal{L}_3. \tag{3.91}$$

(n) *If the constant* β_{-2} *defined in* (3.67) *is zero and the constant* β_{-1} *defined in* (3.68) *is nonzero, then the right primary reflection coefficient* $R(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$R(k) = \frac{\omega_{-1}}{\beta_{-1}} + O(k), \qquad k \to 0 \text{ in } \mathcal{L}_3, \tag{3.92}$$

where ω_{-1} *is the constant defined in* (3.74)*.*

(o) *If the constant* β_{-2} *defined in* (3.67) *and the constant* β_{-1} *defined in* (3.68) *are both zero, then the right primary reflection coefficient* $R(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$R(k) = O(1), \qquad k \to 0 \text{ in } \mathcal{L}_3. \tag{3.93}$$

(p) *If the constant* β_{-2} *defined in* (3.67) *is nonzero, then the right secondary reflection coefficient* $N(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$N(k) = z^2 + O(k), \qquad k \to 0 \text{ in } \mathcal{L}_4. \tag{3.94}$$

(q) *If the constant* β_{-2} *defined in* (3.67) *is zero and the constant* β_{-1} *defined in* (3.68) *is nonzero, then the right secondary reflection coefficient* $N(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$N(k) = \frac{\eta_{-1}}{\beta_{-1}} + O(k), \qquad k \to 0 \text{ in } \mathcal{L}_4, \tag{3.95}$$

where η_{-1} *is the constant defined in* (3.76).

(r) *If the constant* β_{-2} *defined in* (3.67) *and the constant* β_{-1} *defined in* (3.68) *are both zero, then the right secondary reflection coefficient* $N(k)$ *is continuous at* $k = 0$ *and it has the small* k*-asymptotics given by*

$$N(k) = O(1), \qquad k \to 0 \text{ in } \mathcal{L}_4. \tag{3.96}$$

3.6 The bound states for (1.1)

In the description of the scattering theory for (1.1), we have observed that there are six scattering coefficients $T_{\mathrm{l}}(k)$, $T_{\mathrm{r}}(k)$, $L(k)$, $M(k)$, $R(k)$, and $N(k)$ with their k-domains on the directed half lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ in the complex k-plane as shown in Figure 2.1. In particular, the left primary reflection coefficient $L(k)$ is defined when $k \in \mathcal{L}_1$, the right primary reflection coefficient $R(k)$ is defined when $k \in \mathcal{L}_3$, the left secondary reflection coefficient $M(k)$ is defined when $k \in \mathcal{L}_2$, and the right secondary reflection coefficient $N(k)$ is defined when $k \in \mathcal{L}_4$. The left transmission coefficient $T_{\mathrm{l}}(k)$ is originally defined on

$k \in \mathcal{L}_1 \cup \mathcal{L}_2$ and meromorphically extended to $k \in \Omega_1$. Similarly, the right transmission coefficient $T_{\rm r}(k)$ is originally defined on $k \in \mathcal{L}_3 \cup \mathcal{L}_4$ and meromorphically extended to $k \in \Omega_3$. Thus, we can view the scattering for (1.1) taking place on the four half lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ in the complex k-plane.

As we observe from (2.19), a solution to (1.1) vanishing as $x \to +\infty$ must actually vanish exponentially and that a solution vanishing as $x \to -\infty$ must vanish exponentially. A solution to (1.1) that vanishes when $x \to +\infty$ and also vanishes when $x \to -\infty$ does not correspond to a scattering solution to (1.1) but corresponds to a bound-state solution to (1.1). In general, such a solution does not occur at a k-value on the four half lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ if the potentials $Q(x)$ and $P(x)$ satisfy some appropriate restrictions and decay to zero as $x \to \pm\infty$ sufficiently fast. Thus, we expect a bound-state k-value to occur at an interior point in one of the four open sectors Ω_1, Ω_2, Ω_3, Ω_4 shown in Figure 2.1.

In general, a bound state for (1.1) is defined as a nontrivial solution which is square integrable in $x \in \mathbb{R}$. Thus, if there is a bound state at the k-value k_1, then there exists a solution $\Psi(k_1, x)$ to (1.1) at $k = k_1$ so that

$$\int_{-\infty}^{\infty} dx\, |\Psi(k_1, x)|^2 < +\infty. \tag{3.97}$$

We note that if $\Psi(k_1, x)$ is a bound-state wavefunction, then any constant multiple of it is also a bound-state wavefunction. This is because the ODE (1.1) is homogeneous. If there are several bound states for (1.1) occurring at the k-values $k_1, k_2, \ldots, k_{\mathbf{N}}$, then there exists square-integrable solutions $\Psi(k_j, x)$ to (1.1) occurring at $k = k_j$ for $1 \le j \le \mathbf{N}$. We refer to $\mathbf{N}$ as the number of bound states for (1.1).

With each bound state at $k = k_j$ we associate a complex constant D_j, which we refer to as the bound-state dependency constant. The dependency constant relates to each other two fundamentals solutions to (1.1) at $k = k_j$. For example, when k_j belongs to the sector $\Omega_1^{\rm down}$ in the complex k-plane shown in Figure 2.5, the dependency constant D_j occurs as the constant relating the Jost solutions $f(k_j, x)$ and $g(zk_j, x)$ as

$$f(k_j, x) = D_j\, g(zk_j, x), \qquad x \in \mathbb{R}. \tag{3.98}$$

We remark that (3.98) can be derived as follows. As seen from Figure 2.8, at the k-value k_j in Ω_1^{down} there are the three fundamental solutions $f(k_j,x)$, $g(zk_j,x)$, and $h^{\text{up}}(z^2k_j,x)$. We can evaluate their 3-Wronskian by using (2.21), (2.29), (2.35), (2.39), and (3.30). We obtain

$$[f(k_j,x)\,;\,g(zk_j,x)\,;\,h^{\text{up}}(z^2k_j,x)] = -9z^2\,k_j^4\,T_{\text{l}}(k_j)^{-1}\,T_{\text{r}}(zk_j)^{-1}. \tag{3.99}$$

At the bound state $k=k_j$ we have $T_{\text{l}}(k_j)^{-1}=0$, and hence the right-hand side of (3.99) is zero. Consequently, the solutions $f(k_j,x)$, $g(zk_j,x)$, and $h^{\text{up}}(z^2k_j,x)$ are linearly dependent. Thus, there are two constants D_j and E_j so that

$$f(k_j,x) = D_j\,g(zk_j,x) + E_j\,h^{\text{up}}(z^2k_j,x), \qquad x\in\mathbb{R}. \tag{3.100}$$

If we want $f(k_j,x)$ to be a bound-state solution to (1.1), we know that $f(k_j,x)$ must decay exponentially as $x\to+\infty$ and also $x\to-\infty$. Since k_j is located in Ω_1^{down}, from Figure 2.5 we see that the real part of k_j is negative. Hence, from (2.21) we observe that $f(k_j,x)$ decays exponentially as $x\to+\infty$. Thus, we would like the right-hand side of (3.100) to decay exponentially as $x\to-\infty$ so that $f(k_j,x)$ decays exponentially also when $x\to-\infty$. It turns out that $g(zk_j,x)$ decays exponentially as $x\to-\infty$, and this is seen from (2.29) and the fact that the real part of zk_j is positive. On the other hand, $h^{\text{up}}(z^2k_j,x)$ cannot decay exponentially as $x\to-\infty$ and hence we must choose E_j in (3.100) as zero. Thus, (3.100) yields (3.98).

We recall that in the solution to the inverse scattering problem for (1.1), the scattering data set that we use consists of the scattering coefficients for (1.1) and the bound-state information for (1.1). If there are $\mathbf{N}$ bound states for (1.1) at $k=k_j$ with $1\le j\le\mathbf{N}$, then the appropriate bound-state information to use is given by the set $\{k_j,D_j\}_{j=1}^{\mathbf{N}}$, where D_j is the dependency constant at the bound state at $k=k_j$. For the use of bound-state dependency constants in other linear differential equations in inverse scattering theory, we refer the reader to [6,7] and the references therein.

CHAPTER 4

THE INVERSE SCATTERING PROBLEM FOR (1.1)

We recall that the goal in the solution to the inverse scattering problem for (1.1) is the recovery of the potentials $Q(x)$ and $P(x)$ from the scattering data set consisting of the scattering coefficients and the bound-state information. The scattering coefficients for (1.1) consist of the left scattering coefficients $T_{\mathrm{l}}(k)$, $L(k)$, $M(k)$ appearing in (2.35) and the right scattering coefficients $T_{\mathrm{r}}(k)$, $R(k)$, $N(k)$ appearing in (2.39). As we have seen in Section 3.6, the bound-state information consists of the set $\{k_j, D_j\}_{j=1}^{\mathbf{N}}$, where $\mathbf{N}$ denotes the number of bound states, k_j denotes the k-value at which (1.1) has a square-integrable solution, and D_j is the dependency constant at the bound state at $k = k_j$.

In this thesis we consider the analysis of the inverse scattering problem when the secondary reflection coefficients are zero, i.e. when we have

$$M(k) \equiv 0, \qquad k \in \mathcal{L}_2, \tag{4.1}$$

$$N(k) \equiv 0, \qquad k \in \mathcal{L}_4. \tag{4.2}$$

The assumptions in (4.1) and (4.2) enables us to formulate a Riemann–Hilbert problem on the complex k-plane by formulating that problem on the full line $\mathcal{L}$ in $\mathbb{C}$, where we have defined

$$\mathcal{L} := \mathcal{L}_1 \cup (-\mathcal{L}_3). \tag{4.3}$$

We recall that $-\mathcal{L}_3$ is the directed line parametrized by $k = zs$ where $s \in (-\infty, 0]$. The directed full line $\mathcal{L}$ separates the complex k-plane into the two open half planes $\mathcal{P}^+$ and $\mathcal{P}^-$, as shown in Figure 4.1. The half plane $\mathcal{P}^+$ lies to the left of the full line $\mathcal{L}$ and the half plane $\mathcal{P}^-$ lies to the right of $\mathcal{L}$. We use $\overline{\mathcal{P}^+}$ and $\overline{\mathcal{P}^-}$ to denote the closure of $\mathcal{P}^+$ and the closure of $\mathcal{P}^-$, respectively, where we have defined

$$\overline{\mathcal{P}^+} := \mathcal{P}^+ \cup \mathcal{L}, \tag{4.4}$$

$$\overline{\mathcal{P}^-} := \mathcal{P}^- \cup \mathcal{L}. \tag{4.5}$$

We refer to $\mathcal{P}^+$ as the plus region and $\mathcal{P}^-$ as the minus region. We refer to a meromorphic function of k in $\mathcal{P}^+$ as a plus function, and we refer to a meromorphic function of k in $\mathcal{P}^-$ as a minus function.

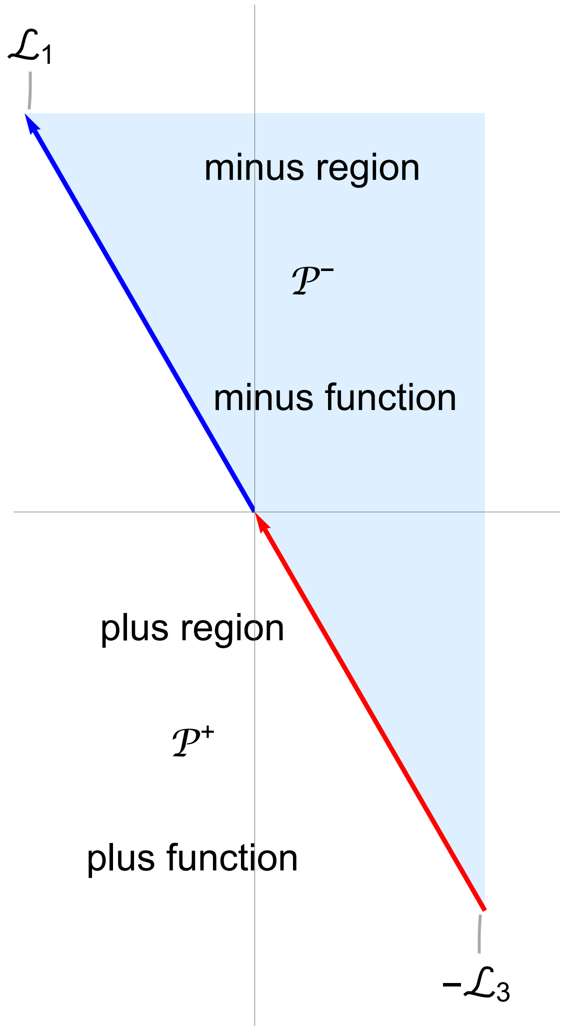

Figure 4.1: The full line $\mathcal{L}$ separating the complex k-plane into the plus region $\mathcal{P}^+$ and the minus region $\mathcal{P}^-$.

For each fixed $x \in \mathbb{R}$, we define the particular function $\Phi_+(k,x)$ as

$$\Phi_+(k,x) := \begin{cases} T_{\rm l}(k)\, f(k,x), & k \in \overline{\Omega}_1, \\ \dfrac{T_{\rm r}(zk)\, h^{\rm down}(k,x)}{z(1-z)k}, & k \in \overline{\Omega}_2, \end{cases} \tag{4.6}$$

where we recall that $f(k,x)$ is the left Jost solution to (1.1) appearing (2.21), $h^{\rm down}(k,x)$ is the solution to (1.1) appearing in (2.57), $T_{\rm l}(k)$ is the left transmission coefficient for (1.1) appearing in (2.35), $T_{\rm r}(k)$ is the right transmission coefficient for (1.1) appearing in (2.39), z is the special constant defined in (2.1), and $\overline{\Omega}_1$ and $\overline{\Omega}_3$ are the closed sectors in the complex k-plane shown in Figure 2.5. In a similar manner, for each fixed $x \in \mathbb{R}$, we define the particular function $\Phi_-(k,x)$ as

$$\Phi_-(k,x) := \begin{cases} \dfrac{T_{\rm r}(z^2k)\, h^{\rm up}(k,x)}{-z(1-z)k}, & k \in \overline{\Omega}_4, \\ g(k,x), & k \in \overline{\Omega}_3, \end{cases} \tag{4.7}$$

where we recall that $g(k,x)$ is the right Jost solution to (1.1) appearing in (2.39), $h^{\rm up}(k,x)$ is the solution to (1.1) appearing in (2.58), and $\overline{\Omega}_3$ and $\overline{\Omega}_4$ are the closed sectors in the complex k-plane shown in Figure 2.5.

In the next theorem, we show that the restriction in (4.1) yields that the two pieces of $\Phi_+(k,x)$ defined in $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively, agree on $k \in \mathcal{L}_2$, and hence the function $\Phi_+(k,x)$ defined in (4.3) becomes a plus function in $k \in \mathcal{P}^+$.

Theorem 4.0.1. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $T_{\rm l}(k)$ and $T_{\rm r}(k)$ be the transmission coefficients for (1.1) appearing in (2.35) and (2.39), respectively. Let $f(k,x)$ and $h^{\rm down}(k,x)$ be the solutions to (1.1) appearing in (2.21) and (2.57), respectively. For each fixed $x \in \mathbb{R}$, we have*

$$T_{\rm l}(k)\, f(k,x) = \frac{T_{\rm r}(zk)\, h^{\rm down}(k,x)}{z(1-z)k}, \qquad k \in \mathcal{L}_2, \tag{4.8}$$

if and only if the left secondary reflection coefficient $M(k)$ is zero, as indicated in (4.1). Consequently, when (4.1) holds the quantity $\Phi_+(k,x)$ defined in (4.3) is a plus function in $k \in \mathcal{P}^+$.

Proof. From Figure 2.8, we see that $f(k,x)$, $h^{\mathrm{down}}(k,x)$ and $h^{\mathrm{up}}(z^2k,x)$ are solutions to (1.1) when $k \in \mathcal{L}_2$. The 3-Wronskian of those three solutions can be evaluated when $k \in \mathcal{L}_2$ by using the spacial asymptotics given in (2.21), (2.35), (3.25), and (3.28) it turns out that that 3-Wronskian is zero. Hence, we can write $f(k,x)$ as a linear combination of $h^{\mathrm{down}}(k,x)$ and $h^{\mathrm{up}}(z^2k,x)$ as

$$f(k,x) = \alpha(k)\, h^{\mathrm{down}}(k,x) + \beta(k)\, h^{\mathrm{up}}(z^2k,x), \qquad k \in \mathcal{L}_2, \quad x \in \mathbb{R}. \tag{4.9}$$

By letting $x \to \pm\infty$ in (4.9) and using (2.21), (2.35), (3.25), and (3.28), we determine coefficients $\alpha(k)$ and $\beta(k)$ as

$$\alpha(k) = \frac{T_{\mathrm{l}}(k)^{-1}}{z(1-z)k\, T_{\mathrm{r}}(zk)^{-1}}, \quad \beta(k) = \frac{-M(k)\, T_{\mathrm{l}}(k)^{-1}}{(1-z)k\, T_{\mathrm{r}}(zk)^{-1}}. \tag{4.10}$$

We see that (4.8) and (4.9) are equivalent if and only if $M(k) \equiv 0$ when $k \in \mathcal{L}_2$. ∎

In a similar manner, in the next theorem we show that the restriction in (4.2) yields that the two pieces of $\Phi_-(k,x)$ defined in $\overline{\Omega}_3$ and $\overline{\Omega}_4$, respectively, agree on $k \in \mathcal{L}_4$, and hence the function $\Phi_-(k,x)$ defined in (4.7) becomes a minus function in $k \in \mathcal{P}^-$.

Theorem 4.0.2. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Let $T_{\mathrm{r}}(k)$ be the transmission coefficient for (1.1) appearing in (2.39), and let $g(k,x)$ and $h^{\mathrm{up}}(k,x)$ be the solutions to (1.1) appearing in (2.29) and (2.58), respectively. For each fixed $x \in \mathbb{R}$, we have*

$$g(k,x) = \frac{T_{\mathrm{r}}(z^2k)\, h^{\mathrm{up}}(k,x)}{-z(1-z)k}, \qquad k \in \mathcal{L}_4, \quad x \in \mathbb{R}. \tag{4.11}$$

if and only if the right secondary reflection coefficient $N(k)$ is zero, as indicated in (4.2). Consequently, when (4.2) holds the quantity $\Phi_-(k,x)$ defined in (4.7) is a minus function in $k \in \mathcal{P}^-$.

Proof. The proof is similar to the proof of Theorem 4.0.1. From Figure 2.8, we observe that $g(k,x)$, $h^{\mathrm{up}}(k,x)$, and $h^{\mathrm{down}}(z^2k,x)$ are solutions to (1.1) when $k \in \mathcal{L}_4$. Using the spacial asymptotics in (2.30), (2.39), (3.26), and (3.29), we evaluate the 3-Wronskian of those solutions and find that that Wronskian is zero when $k \in \mathcal{L}_4$. Thus, we can express $g(k,x)$ as a linear combination of $h^{\rm up}(k,x)$ and $h^{\rm down}(z^2k,x)$ as

$$g(k,x) = \gamma(k)\, h^{\rm up}(k,x) + \epsilon(k)\, h^{\rm down}(z^2k,x), \qquad k \in \mathcal{L}_4, \quad x \in \mathbb{R}. \tag{4.12}$$

By letting $x \to \pm\infty$ in (4.12) and using (2.29), (2.39), (3.26), and (3.29), we obtain the coefficients $\gamma(k)$ and $\epsilon(k)$ as

$$\gamma(k) = \frac{-T_{\rm r}(z^2k)^{-1}}{z(1-z)k\,[1-R(z^2k)\,N(k)]}, \quad \epsilon(k) = \frac{N(k)\,T_{\rm r}(k)^{-1}}{(1-z)k\,T_{\rm l}(zk)^{-1}}, \tag{4.13}$$

where we have also used (3.36) to obtain $\gamma(k)$. We see that (4.11) and (4.12) are equivalent if and only if $N(k) \equiv 0$ when $k \in \mathcal{L}_4$. ■

In the next theorem, we present some relevant properties of $\Phi_+(k,x)$ and $\Phi_-(k,x)$ defined in (4.3) and (4.4), respectively.

Theorem 4.0.3. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Further, assume that the secondary reflection coefficients $M(k)$ and $N(k)$ for (1.1) are both zero, i.e. that (4.1) and (4.2) hold. We then have the following:*

(a) *For each fixed $x \in \mathbb{R}$, the quantity $\Phi_+(k,x)$ is continuous on $k \in \mathcal{L}$.*

(b) *For each fixed $x \in \mathbb{R}$, the quantity $\Phi_-(k,x)$ is continuous on $k \in \mathcal{L}$.*

(c) *The function $\Phi_+(k,x)$ is a solution to (1.1) and meromorphic in $k \in \mathcal{P}^+$. Similarly, the function $\Phi_-(k,x)$ is a solution to (1.1) and meromorphic in $k \in \mathcal{P}^-$.*

(d) *For each fixed $x \in \mathbb{R}$, the asymptotics of $\Phi_+(k,x)$ as $k \to \infty$ in $\overline{\mathcal{P}^+}$ is given by*

$$\Phi_+(k,x) = e^{kx}\left[1 + O\left(\frac{1}{k}\right)\right]. \tag{4.14}$$

(e) *For each fixed $x \in \mathbb{R}$, the asymptotics of $\Phi_-(k,x)$ as $k \to \infty$ in $\overline{\mathcal{P}^-}$ is given by*

$$\Phi_-(k,x) = e^{kx}\left[1 + O\left(\frac{1}{k}\right)\right]. \tag{4.15}$$

(f) *Consequently, for each fixed $x \in \mathbb{R}$, each of the quantities $e^{-kx}\,[\Phi_+(k,x) - 1]$ and $e^{-kx}\,[\Phi_-(k,x) - 1]$ is a square-integrable function of k when $k \in \mathcal{L}$.*

Proof. The proof of (a) is given as follows. From (4.6) we see that $\Phi_+(k,x)$ is continuous in k for $k \in \mathcal{L}$ provided $f(k,x)$ and $T_{\mathrm{l}}(k)$ are continuous in k for $k \in \mathcal{L}_1$ and that $h^{\mathrm{down}}(k,x)$ and $T_{\mathrm{r}}(zk)/k$ are continuous in k for $k \in \mathcal{L}_3$. The continuity of $f(k,x)$ follows from Theorem 2.10.1(a), the continuity of $h^{\mathrm{down}}(k,x)$ follows from Theorem 2.10.1(c), and the continuity properties of $T_{\mathrm{l}}(k)$ and $T_{\mathrm{r}}(zk)/k$ follow from Theorem 2.10.2 and Corollary 3.5.1.1. Hence, the proof of (a) is complete. For the proof of (b) we proceed as follows. From (4.7) we see that $\Phi_-(k,x)$ is continuous in k for $k \in \mathcal{L}$ provided $g(k,x)$ is continuous in k for $k \in \mathcal{L}_3$ and that $h^{\mathrm{up}}(k,x)$ and $T_{\mathrm{r}}(z^2k)/k$ are continuous in k for $k \in \mathcal{L}_4$. The continuity of $g(k,x)$ follows from Theorem 2.10.1(b), the continuity of $h^{\mathrm{up}}(k,x)$ follows from Theorem 2.10.1(d), and the continuity of $T_{\mathrm{r}}(z^2k)/k$ follows from Theorem 2.10.2 and Corollary 3.5.1.1. Thus, the proof of (b) is complete. The proof of (c) is obtained as follows. Because $f(k,x)$ and $h^{\mathrm{down}}(k,x)$ are solutions to (1.1), from (4.6) we see that $\Phi_+(k,x)$ is also a solution to (1.1). The meromorphic property of $\Phi_+(k,x)$ in $k \in \mathcal{P}^+$ follows from Theorem 4.0.1. Similarly, since $g(k,x)$ and $h^{\mathrm{up}}(k,x)$ are solutions to (1.1), from (4.7) it follows that $\Phi_-(k,x)$ is also a solution to (1.1). From Theorem 4.0.2 it follows that $\Phi_-(k,x)$ is meromorphic in $k \in \mathcal{P}^-$. Hence, the proof of (c) is complete. For the proof of (d), we proceed as follows. We obtain the large k-asymptotics of $\Phi_+(k,x)$ given in (4.24) by using the large k-asymptotics of $f(k,x)$, $h^{\mathrm{down}}(k,x)$, $T_{\mathrm{l}}(k)$, and $T_{\mathrm{r}}(k)$ given in (3.40), (3.45), (3.54), and (3.58), respectively. This completes the proof of (d). In order to establish the large k-asymptotics of $\Phi_-(k,x)$ given in (4.15), we use the large k-asymptotics of $g(k,x)$, $h^{\mathrm{up}}(k,x)$, and $T_{\mathrm{r}}(k)$ given in (3.44), (3.48), and (3.58), respectively. This completes the proof of (e). The result of (f) directly follows from (d) and (e). ■

In Theorem 4.0.3, we have established that the quantity $\Phi_+(k,x)$ is a solution to (1.1) and is a plus function in $k \in \mathcal{P}^+$ and that the quantity $\Phi_-(k,x)$ is a solution to (1.1) and is a minus function in $k \in \mathcal{P}^-$. We also know that $\Phi_+(k,x)$ and $\Phi_-(k,x)$ have their common k-domain given by the full line $\mathcal{L}$ in $\mathbb{C}$. However, in general those two functions do not agree when $k \in \mathcal{L}$. We refer to the difference $\Phi_+(k,x) - \Phi_-(k,x)$ as the jump and denote it by $J(k,x)$, i.e. we let

$$J(k,x) := \Phi_+(k,x) - \Phi_-(k,x), \qquad k \in \mathcal{L}, \quad x \in \mathbb{R}. \tag{4.16}$$

In the next theorem, we express $J(k,x)$ defined in (4.16) in terms of the primary reflection coefficients $L(k)$ and $R(k)$ for (1.1), under the assumption that the secondary reflection coefficients $M(k)$ and $N(k)$ are zero.

Theorem 4.0.4. *Assume that the potentials $Q(x)$ and $P(x)$ in (1.1) belong to the Schwartz class $\mathcal{S}(\mathbb{R})$. Further, assume that the secondary reflections $M(k)$ and $N(k)$ for (1.1) are both zero, i.e. that (4.1) and (4.2) hold. Then, the jump $J(k,x)$ defined in (4.16) is given by*

$$J(k,x) = \begin{cases} L(k)\, T_{\rm l}(zk)\, f(zk,x), & k \in \mathcal{L}_1, \\ -R(k)\, \dfrac{T_{\rm r}(zk)}{T_{\rm r}(k)}\, g(zk,x), & k \in -\mathcal{L}_3, \end{cases} \tag{4.17}$$

where we recall that $T_{\rm l}(k)$ and $L(k)$ are the left scattering coefficients for (1.1) appearing in (2.35), $T_{\rm r}(k)$ and $R(k)$ are the right scattering coefficients for (1.1) appearing in (2.39), and $f(k,x)$ and $g(k,x)$ are the Jost solutions to (1.1) appearing in (2.21) and (2.29), respectively.

Proof. The jump value stated in the first line on the right-hand side of (4.17) is obtained as follows. From Figure 2.8 we see that $f(k,x)$, $h^{\rm up}(k,x)$, $f(zk,x)$ are solutions to (1.1) when $k \in \mathcal{L}_1$. We evaluate the 3-Wronskian of those three solutions with the help of the spacial asymptotics in (2.21), (2.35), and (3.28). It turns out that that 3-Wronskian is zero when $k \in \mathcal{L}_1$. Thus, we can express $f(k,x)$ as a linear combination of $h^{\rm up}(k,x)$ and $f(zk,x)$ as

$$f(k,x) = \omega(k)\, h^{\rm up}(k,x) + \eta(k)\, f(zk,x), \qquad k \in \mathcal{L}_1, \quad x \in \mathbb{R}. \tag{4.18}$$

By letting $x \to \pm\infty$ in (4.18), with the help of the spacial asymptotics in (2.21), (2.35), and (3.28) we obtain the coefficients $\omega(k)$ and $\eta(k)$ as

$$\omega(k) = \frac{-T_{\rm l}(k)^{-1}[1 - L(k)\, M(zk)]}{z(1-z)k\, T_{\rm r}(z^2k)^{-1}}, \quad \eta(k) = \frac{L(k)\, T_{\rm l}(k)^{-1}}{T_{\rm l}(zk)^{-1}}, \tag{4.19}$$

where we have also used (3.35) to obtain $\omega(k)$. Let us write (4.18) in the equivalent form

$$T_{\rm l}(k)\, f(k,x) = T_{\rm l}(k)\, \omega(k)\, h^{\rm up}(k,x) + T_{\rm l}(k)\, \eta(k)\, f(zk,x), \qquad k \in \mathcal{L}_1, \quad x \in \mathbb{R}. \tag{4.20}$$

When $M(k) \equiv 0$ for $k \in \mathcal{L}_2$, the equality in (4.20) yields the jump $J(k,x)$ for $k \in \mathcal{L}_1$ given in (4.17).

Similarly, we obtain the jump value stated in the second line on the right-hand side of (4.17) as follows. From Figure 2.8 we see that $g(k,x)$, $h^{\rm down}(k,x)$, and $g(zk,x)$ are solutions to (1.1) when $k \in \mathcal{L}_3$. With the help of the spacial asymptotics in (2.29), (2.39), (3.26), we see that the 3-Wronskian of $g(k,x)$, $h^{\rm down}(k,x)$, and $g(zk,x)$ is zero when $k \in \mathcal{L}_3$. Thus, we can express $g(k,x)$ as a linear combination of $h^{\rm down}(k,x)$ and $g(zk,x)$ as

$$g(k,x) = \xi(k)\, h^{\rm down}(k,x) + \zeta(k)\, g(zk,x), \qquad k \in \mathcal{L}_3, \quad x \in \mathbb{R}. \tag{4.21}$$

By letting $x \to \pm\infty$ in (4.21), with the help of (2.29), (2.39), (3.26), we determine the coefficients $\xi(k)$ and $\zeta(k)$ as

$$\xi(k) = \frac{T_{\rm r}(zk)}{z(1-z)k}, \quad \zeta(k) = \frac{R(k)\, T_{\rm r}(zk)}{T_{\rm r}(k)}, \tag{4.22}$$

where we have also used (3.36) to obtain $\xi(k)$ and assumed that $N(k) \equiv 0$ for $k \in \mathcal{L}_4$. By rewriting (4.21) as

$$\xi(k)\, h^{\rm down}(k,x) = g(k,x) - \zeta(k)\, g(zk,x), \qquad k \in \mathcal{L}_3, \quad x \in \mathbb{R}, \tag{4.23}$$

we see that the equality in (4.23) yields the jump $J(k,x)$ for $k \in \mathcal{L}_3$ given in (4.17). ∎

4.1 The Riemann–Hilbert problem for (1.1)

As a consequence of Theorems 4.0.3 and 4.0.4, we obtain the Riemann–Hilbert problem for (1.1) given by

$$\Phi_+(k,x) = \Phi_-(k,x) + J(k,x), \qquad k \in \mathcal{L}, \tag{4.24}$$

where $\mathcal{L}$ is the full line in the complex k-plane separating the plus region $\mathcal{P}^+$ and the minus region $\mathcal{P}^-$, the quantity $\Phi_+(k,x)$ is the plus function defined in (4.6), the quantity $\Phi_-(k,x)$

is the minus function given in (4.7), and $J(k,x)$ is the jump function given in (4.17). We recall that (4.24) is valid when the secondary reflection coefficients $M(k)$ and $N(k)$ for (1.1) are both zero. We also recall that each of $\Phi_+(k,x)$ and $\Phi_-(k,x)$ is a solution to (1.1), and their common k-domain is the full line $\mathcal{L}$.

The solution to the Riemann-Hilbert problem (4.24) yields the solution to the inverse scattering problem for (1.1) as follows. For simplicity, we describe the solution to the Riemann–Hilbert problem (4.24) when there are no bound states for (1.1). In that case, given the four scattering coefficients $T_{\rm l}(k)$, $L(k)$, $T_{\rm r}(k)$, and $R(k)$ for (1.1), we solve the Riemann–Hilbert problem (4.24) by determining $\Phi_+(k,x)$ and $\Phi_-(k,x)$ in $\overline{\mathcal{P}^+}$ and $\overline{\mathcal{P}^-}$, respectively, for each fixed $x \in \mathbb{R}$. Once we obtain $\Phi_+(k,x)$ and $\Phi_-(k,x)$, from (4.6) we see that we have recovered the left Jost solution $f(k,x)$ in $\overline{\Omega}_1$ and from (4.7) we see that we have recovered the right Jost solution $g(k,x)$ in $\overline{\Omega}_3$. Then, as seen from (2.22) and (2.30), we have also recovered $u(k,x)$ and $v(k,x)$ appearing in (3.37) and (3.41), respectively. Finally, we can recover the two potentials $Q(x)$ and $P(x)$ in (1.1) from either of the quantities $u(k,x)$ and $v(k,x)$. For example, the recovery of $Q(x)$ and $P(x)$ from $u(k,x)$ is accomplished as follows. Having $u(k,x)$ at hand, we see from (3.37) that we also have $u_1(x)$ and $u_2(x)$ defined in (3.38) and (3.39), respectively. From (3.38), we get the potential $Q(x)$ as

$$Q(x) = -3\, u_1'(x), \qquad x \in \mathbb{R}, \tag{4.25}$$

which is obtained by differentating both sides of (3.38). By taking the x-derivative of both sides of (3.39), we obtain

$$u_2'(x) = \frac{1}{3}[Q'(x) - P(x)] - \frac{1}{9}\, Q(x) \int_x^\infty dy\, Q(y), \qquad x \in \mathbb{R}. \tag{4.26}$$

Using (4.25) in (4.26), we recover the potential $P(x)$ as

$$P(x) = 3\left[u_1(x)\, u_1'(x) - u_1''(x) - u_2'(x)\right], \qquad x \in \mathbb{R}. \tag{4.27}$$

The recovery of the potentials $Q(x)$ and $P(x)$ from the quantity $v(k,x)$ is accomplished in a similar manner. Having $v(k,x)$ at hand, we see from (3.41) that we also have $v_1(x)$ and

$v_2(x)$ defined in (3.42) and (3.43), respectively. By taking the x-derivative of both sides of (3.42), we obtain the potential $Q(x)$ as

$$Q(x) = -3\, v_1'(x), \qquad x \in \mathbb{R}. \tag{4.28}$$

By taking the x-derivative of both sides of (3.43), we obtain

$$v_2'(x) = \frac{1}{3}[Q'(x) - P(x)] + \frac{1}{9}\, Q(x) \int_{-\infty}^{x} dy\, Q(y), \qquad x \in \mathbb{R}. \tag{4.29}$$

Using (4.28) in (4.29), we recover the potential $P(x)$ as

$$P(x) = 3\left[v_1(x)\, v_1'(x) - v_1''(x) - v_2'(x)\right], \qquad x \in \mathbb{R}. \tag{4.30}$$

We conclude the discussion of the inverse scattering problem for (1.1) with the following remark. As seen from Theorem 4.0.3(f), for each fixed $x \in \mathbb{R}$, the modified plus function $e^{-kx}[\Phi_+(k,x) - 1]$ and the modified minus function $e^{-kx}[\Phi_-(k,x) - 1]$ are each square integrable in $k \in \mathcal{L}$ in the complex k-plane. Hence, we can take Fourier transformations of those modified functions for $k \in \mathcal{L}$. Let us write the Riemann–Hilbert problem (4.24) as

$$e^{-kx}[\Phi_+(k,x) - 1] = e^{-kx}[\Phi_-(k,x) - 1] + e^{-kx}\, J(k,x), \qquad k \in \mathcal{L}. \tag{4.31}$$

From (4.31) we observe that the quantity $e^{-kx}J(k,x)$ is also square integrable when $k \in \mathcal{L}$. By using the Fourier transformation on each side of (4.31) for $k \in \mathcal{L}$, we are able to transform the modified Riemann–Hilbert problem in (4.31) into a linear integral equation, which we refer to as the Marchenko integral equation. In particular, we define the quantity $K(x,y)$, which is the Fourier transform for $e^{-kx}[\Phi_+(k,x) - 1]$ for $k \in \mathcal{L}$, as

$$K(x,y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds\, e^{isy}\, e^{-zsx}[\Phi_+(zs,x) - 1], \qquad x \in \mathbb{R}, \tag{4.32}$$

where we recall that z is the special constant appearing in (2.1) and that the directed full line $\mathcal{L}$ is parametrized as $k = zs$ with $s \in (-\infty, +\infty)$. For each fixed $x \in \mathbb{R}$, the quantity $K(x,y)$ satisfies an integral equation in $y \in \mathbb{R}$, which is our Marchenko integral equation.

It is possible to rewrite that Marchenko integral equation in an equivalent form as a system of two integral equations for $y > 0$ and for $y < 0$, respectively. We refer to that system of two integral equations as the Marchenko system of integral equations. The potentials $Q(x)$ and $P(x)$ in (1.1) can be obtained from the solution $K(x, y)$ to that Marchenko system. This procedure is the analog of the Marchenko method [13, 16, 19, 30, 31, 35] used to solve the inverse scattering problem for the full-line Schrödinger equation. The details of our Marchenko method for (1.1) and the recovery of the two potentials $Q(x)$ and $P(x)$ from the solution to our Marchenko system for (1.1) will be published elsewhere.

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BIOGRAPHICAL STATEMENT

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