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Novel Roles of Standard and Non-standard Null Lagrangians in Classical and Quantum Physics

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NOVEL ROLES OF STANDARD AND NON-STANDARD NULL LAGRANGIANS IN CLASSICAL AND QUANTUM PHYSICS

by

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ABSTRACT

NOVEL ROLES OF STANDARD AND NON-STANDARD NULL LAGRANGIANS IN CLASSICAL AND QUANTUM PHYSICS

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The three known families of Lagrangians are standard, non-standard, and null Lagrangians. While Lagrangians are widely used in physics for their ability to characterize physical systems, most of this work uses only standard Lagrangians. As such, standard Lagrangians have been studied in physics for around three hundred years; however, non-standard Lagrangians for systems in physics have been considered only for a few decades, and null have been ignored almost entirely. Although only some Lagrangians are null Lagrangians, all Lagrangians can be categorized as either standard or non-standard Lagrangians, meaning that there exist standard null Lagrangians and non-standard null Lagrangians. My dissertation is devoted to the study of null Lagrangians in physics, with some additional applications of non-standard Lagrangians.

Non-standard Lagrangians, which are Lagrangians with forms different from standard Lagrangians, have been studied in physics, but significantly less so than standard Lagrangians. This family of Lagrangians is known for having indiscernible kinetic and potential energy terms. However, it is shown herein that the non-standard

Lagrangian for the Law of Inertia preserves its Galilean invariance, which is notably different from the standard Lagrangian formulation.

Null Lagrangians are a special family of Lagrangians known for yielding identically zero from the Euler-Lagrange equation, and, in this way, not contributing to the equation of motion. Although this special but lesser-known family of Lagrangians has been studied in Mathematics since the 1960s, very little work has been done concerning them outside of this field. Though it might seem that as a result they would be of little use for physical systems, this could not be farther from the truth. The work presented in this dissertation comprises what I hope will become the first step in a larger body of work exploring the role of null and non-standard Lagrangians for physical systems. To the best of our knowledge, this is the first PhD dissertation investigating null Lagrangians in physics.

I show how the addition of a null Lagrangian is sufficient to introduce force to a system, converting an undriven system to a driven one. In this way, forces naturally arise out of the gauge terms corresponding to these null Lagrangians. A formalism for constructing null Lagrangians for systems in dynamics, along with a generalized approach showing how null Lagrangians and their gauge functions can be linked to known forces, is developed and presented. Further, this formalism for introducing forces is extended to dissipative systems through an application to the Bateman oscillator system. It is then shown how nonlinearities can also be introduced through null Lagrangians, including the Duffing oscillator.

Compelling results from the application of null Lagrangians to physical systems of increasing complexity are presented and discussed. Particular attention is given to applications to oscillators, including the Bateman oscillator system, so as to illustrate the physical implications of the work. Null Lagrangians are found for equations with special function solutions in mathematical physics, including Bessel functions, and

Legendre and Hermite polynomials. The connection between a system's Galilean invariance and how it behaves under the introduction of specific gauge functions corresponding to null Lagrangians is discussed and explored for multiple equations of key interest in physics.

As this work is not constrained to classical dynamics, I then show how to formulate physically consistent null Lagrangians for systems in quantum mechanics. The Galilean invariance of the Schrödinger Lagrangian is investigated. Null Lagrangians of a similar form to the Schrödinger Lagrangian are presented and whether they can be used to replace the phase factor required for the aforementioned Galilean invariance is discussed.

As we seek to better understand our universe, null Lagrangians have the potential to be a powerful new lens with which to view and investigate physical phenomena; what underlying symmetries might be uncovered using these new tools? The work presented in this dissertation shows that the investigation of null Lagrangians for systems in physics has already yielded exciting results, and that this promising area of research should be further explored.

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CHAPTER 1

Introduction

In modern physics, the equations of motion, which describe the time and space evolution of different physical systems, are obtained by using the laws of physics. The equation of motion for a given physical system can also be derived by using the so-called Lagrangian formalism; this is the primary topic of research of this dissertation. The formalism was originally developed by Euler (1744) and Lagrange (1788). This effective and elegant procedure is commonly used in classical and quantum mechanics and is based on the principle of least action. This principle was first formulated by Maupertuis (1742) and its modern form was proposed nearly a century later by Hamilton (1834). The principle requires a function that is now known as a Lagrangian, the knowledge of which is necessary to obtain an equation of motion. Different equations of motion require different Lagrangians, which means that they must be known before the equations of motion are derived. As no theory exists as of yet that allows for the construction of Lagrangians from first principles, the forms of such Lagrangians are often initially found simply by guessing the necessary form such that they yield the equations of motion, which are already known, for given dynamical systems.

The first attempts at finding a formalism to replace this method of 'guessing' Lagrangians for a given system by their actual derivations dates back to Helmholtz (1886), who formulated special conditions for the existence of Lagrangians for a general form of second-order differential equations; these include those known in physics. Over the years, a number of different methods of deriving Lagrangians for given equations of motion have been developed and detailed references to such work can be found in the following chapters of this dissertation. It is well known that Lagrangians are not unique and that different Lagrangians may give the same equation of motion.

Typically, Lagrangians are classified as standard and non-standard. In standard Lagrangians, which are very common in physics, the kinetic and potential energy-like terms can easily be identified. Non-standard, however, have indistinguishable kinetic and potential energy terms. Nevertheless, despite having very different mathematical and physical forms, both standard and non-standard Lagrangians they give the same equations of motion; both of these types of Lagrangians are investigated in this dissertation. Further, an additional family of Lagrangians is that of null Lagrangians, whose main characteristic is that their addition to a standard or non-standard Lagrangian makes no difference in the resulting equation of motion. This could be taken to imply that null Lagrangians are of no significance in physics. Thus, the main aim of this dissertation is to demonstrate the novel role of null Lagrangians in non-relativistic classical and quantum physics.

This dissertation is organized as follows: Chapter 2 presents the background for my research by reviewing the Lagrangian formalism along with a brief look at the historical context of Lagrangians. This chapter introduces the three families of Lagrangians I will discuss: standard, non-standard, and null Lagrangians. In Chapter 3, I illustrate the difference in standard and non-standard Lagrangians; special attention is given to the Law of Inertia. The Bateman oscillators are also introduced and discussed.

Chapters 4 through 6 contain my work with Null Lagrangians. In Chapter 4, I begin with methods to construct null Lagrangians, including a generalized approach; further, I show how forces naturally arise out of the gauge terms corresponding to these null Lagrangians. I then present a compelling result from an application of standard null Lagrangians to physical systems of increasing complexity. Particular attention is given to applications to oscillators, including the Bateman oscillator system, to illustrate the physical implications of the work. I also introduce a connection to nonlinearities and show that nonlinear terms can be formulated in a similar way to what was done for forces.

Chapter 5 expands the formalism presented in the chapter preceding it and extends this work to non-standard and non-standard null Lagrangians. An application of the null Lagrangian formalism to special functions of key interest in physics is also presented. The introduction of Galilean invariance by way of the gauge functions associated with null Lagrangians is then discussed. In Chapter 6, I show how to construct null Lagrangians for quantum fields. The Galilean invariance of the Schrödinger equation is investigated, along with that of the Schrödinger Lagrangian, and the associated phase factor is discussed.

Chapter 7 is a summary of key points and findings of my research. Conclusions are drawn and I explore the implications of these results. Further, suggestions are made for possible future work in this promising area of physics.

CHAPTER 2

Background

2.1 Brief Historical Remarks and Motivation

The basic concepts of the calculus of variations were originally established by Euler (1744). Euler was motivated by work done a couple years prior by de Maupertuis (1742), who is credited with formulating the principle of least action around the same time. Notably, some historians point out that Leibniz had already presented the first version of this principle around 1707. Lagrange (1788) then refined Euler's approach and used it to demonstrate that there is a function, which is now known as a Lagrangian, whose prior knowledge is sufficient to derive the equations of Newtonian dynamics. Lagrange's results were published in his work, *Analytical Mechanics*, first in 1788. The principle of least action as formulated by Lagrange applies only to virtual paths whose energy is the same as the real path. This limitation was then removed by William R. Hamilton in 1834, who formulated his version of the principle that is now known as *Hamilton's Principle*.

Hamilton's Principle can be stated as follows: 'A mechanical system moves from one configuration to another in such a way that the variation of the integral of the Lagrangian over time between the path taken and a neighboring virtual path, coterminous in space and time with the actual path, is zero.' This means that a function called a Lagrangian must be specified for the principle to be applied. The existence of this function for second-order ordinary differential equations (ODEs) is guaranteed by a set of conditions proposed by Helmholtz (1886); the conditions can also be used to derive Lagrangians for such ODEs. For the next hundred years, most Lagrangians considered were standard, in reference to the original Lagrangian for Newton's laws of dynamics, introduced first by Lagrange (1788). Note that standard Lagrangians, as aforementioned, are characterized by the difference between the kinetic and potential energy terms.

Standard Lagrangians are also sometimes called natural Lagrangians. Further, Arnold (1978) is credited with first introducing non-natural Lagrangians, which do not have terms that have clearly discernible energy-like forms but yield the same equations of motion. Non-natural Lagrangians are now more commonly known as non-standard Lagrangians and this is the name that will be used to refer to them throughout this dissertation. Null Lagrangians, also known as trivial Lagrangians, have been studied in mathematics since early sixties (e.g., Edelen 1962; Eriksen 1962) and still remain an active area of research in mathematics. Null Lagrangians, another lesser known family of Lagrangians, were virtually ignored in physics with the exception of elasticity, where null Lagrangians were associated with the energy density function of materials (e.g., Anderson et al., 1999). A more detailed description of non-standard and null Lagrangians is given in Section 2.3.

The primary motivation for performing the research presented in this dissertation was to gain new insight into the physical meaning of null Lagrangians and to use this insight to explore the different ways that these Lagrangians may contribute to classical and quantum physics. Because of this strong physical motivation, the described research and presented results are novel in physics and significantly different than those obtained previously by mathematicians. Rather, the main emphasis of this dissertation is on the physical aspects of null Lagrangians and their applications to physical systems as well as implications for these systems. To the best of our knowledge, this is the first PhD dissertation in physics fully devoted to null Lagrangians and their novel roles in classical and quantum mechanics.

2.2 Lagrangian Formalism and Helmholtz Conditions

The Lagrange formalism deals with a functional $\mathcal{A}[x(t)]$, where A is the action and $x(t)$ is an ordinary (with the maps $x : \mathcal{R} \to \mathcal{R}$, with $\mathcal{R}$ denoting the real numbers) and smooth (with at least two continuous derivatives $\mathcal{C}^2$) function to be determined. Typically, $\mathcal{A}[x(t)]$ is given by an integral over a smooth function $L(\dot{x}, x, t)$ that is called a Lagrangian and $\dot{x}$, which is a derivative of x with respect to t. The integral as defined in this way is a mathematical representation of Hamilton's Principle (Hamilton, 1834, Goldstein et al., 2002), which requires that $\delta\mathcal{A} = 0$, where δ is the variation known also as the functional (or Fréchet) derivative of $\mathcal{A}[x(t)]$ with respect to $x(t)$. Using $\delta\mathcal{A} = 0$, the Euler-Lagrange (E-L) equation is obtained, and this equation is a necessary condition for the action to be stationary (to have either a minimum or maximum or saddle point).

Let $\hat{EL}$ be the Euler-Lagrange operator defined as

$$\hat{EL} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}} - \frac{\partial}{\partial x} \ . \tag{2.1}$$

The E-L equation then becomes $\hat{EL}[L(\dot{x}, x, t)] = 0$. In general, this equation gives a second-order ODE that can be further solved to obtain $x(t)$ that makes the action stationary. This procedure of deriving the second-order ODE from the E-L equation is known as the *Lagrangian formalism*; notably, this formalism requires prior knowledge of $L(\dot{x}, x, t)$, either as a standard or non-standard form as both give the same equation of motion. To fully establish the Lagrangian formalism for ordinary differential equations (ODEs), methods to construct standard or non-standard Lagrangians for a given ODE must be developed (see Section 2.3). The construction of Lagrangians for given ODEs is called the inverse calculus of variations problem, also referred to as the Helmholtz problem.

The existence of Lagrangians is guaranteed by the Helmholtz conditions (Helmholtz, 1887; Lopuszanski, 1999), which are necessary and sufficient conditions. Let $F_i(\ddot{x}_j, \dot{x}_j, x_j, t) = 0$ be a set of n ODEs, with $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$. The Helmholtz conditions are then

$$\frac{\partial F_i}{\partial \ddot{x}_j} = \frac{\partial F_j}{\partial \ddot{x}_i} \ , \tag{2.2}$$

$$\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} = \frac{1}{2} \frac{d}{dt} \left(\frac{\partial F_i}{\partial \dot{x}_j} - \frac{\partial F_j}{\partial \dot{x}_i} \right) \ , \tag{2.3}$$

and

$$\frac{\partial F_i}{\partial \dot{x}_j} + \frac{\partial F_j}{\partial \dot{x}_i} = 2 \frac{d}{dt} \left(\frac{\partial F_j}{\partial \ddot{x}_i} \right) \ . \tag{2.4}$$

Further, the conditions can be used to verify whether or not a Lagrangian exists for a given ODE. For the case that it does exist, the Lagrangian can then be found from these conditions. Notably, all standard Lagrangians satisfy these conditions. The same should be true for non-standard Lagrangians; however, there are some exceptions (e.g., Musielak et al., 2020). Note that null Lagrangians identically satisfy the Helmholtz conditions, which means that they cannot be determined by these conditions. Therefore, null Lagrangians can be added to any standard or non-standard Lagrangian without changing the original equation or affecting the Helmholtz conditions.

The Lagrangian formalism requires prior knowledge of a Lagrangian. In general, there are no first principle methods to obtain Lagrangians, which are typically presented without any explanation as to their origin. In physics, most equations of motion were established first and only then were their Lagrangians found, which was often done simply by guessing. Once the Lagrangians are known, the process of finding the resulting dynamical equations is straightforward and can be done by substituting the given Lagrangians into the E-L equation.

2.3 Families of Lagrangians

2.3.1 Standard Lagrangians

Lagrangians are widely used in physics to characterize physical systems and obtain their equations of motion. As originally shown by Lagrange (see Lagrange 1997), Lagrangians for one-dimensional dynamical systems represent the difference between the kinetic and potential energy of these systems, and they can be written as $L(\dot{x}, x) = \dot{x}^2/2 - V(x)$, with $V(x)$ being the potential energy; note that the energies are given per unit mass. It is common to call such Lagrangians standard.

The main characteristic of standard Lagrangians is that their kinetic and potential energy terms can be easily identified. Following Lagrange, in standard Lagrangians the potential energy $E_{pot}(x)$ is subtracted from the kinetic energy, $E_{kin}(\dot{x})$. Rather, $L(\dot{x}, x) = E_{kin}(\dot{x}) - E(x)$. The equation of motion resulting from this Lagrangian is derived from $\hat{EL}[L(\dot{x}, x)] = 0$; it is a second-order ODE. Further, this obtained equation of motion describes the time evolution of the system whose Lagrangian is $L(\dot{x}, x)$. Physical systems with well-defined Lagrangians must be conservative, as non-conservative systems require significant modifications of standard Lagrangians, which make these Lagrangians non-standard; this transition from standard to non-standard Lagrangians is explored in this dissertation, and specific physical examples are presented and discussed in Chapter 3.

Standard Lagrangians are well-known for conservative dynamical systems of classical mechanics (CM) such as the law of inertia, undriven and undamped harmonic oscillators, a linear, undamped pendulum, and other systems. There has also been some progress in deriving standard and non-standard Lagrangians for physical systems described by different ODEs (e.g., Helmholtz 1887; Douglas 1941; Hojman 1984, 1992; Musielak 2008; Cieslinski & Nikiciuk 2010; Musielak et al. 2020a, 2020b). Further, there are standard Lagrangians for quantum systems described by the Schrödinger equation (e.g., Doughty 1990), which can be derived from the Lagrangian formalism that deals with fields and wave functions instead of classical particles (see Chapter 6).

2.3.2 Non-standard Lagrangians

Unlike standard Lagrangians, non-standard Lagrangians (NSLs) are Lagrangians whose potential and kinetic energy terms are not easily distinguishable. The role of standard Lagrangians, which have kinetic and potential energy-like terms that can easily be identified, has been well established in classical mechanics (e.g., Lagrange 1997; Goldstein et al. 2002; Jose & Saletan 2002). Notably, so-called non-standard Lagrangians have only been introduced to CM in recent years (e.g., Arnold 1978; Chandrasekhar et al. 2005; Carinena et al. 2005; Nucci & Leach 2007, 2008a, 2008b; Musielak 2008, 2009; Cieslinski & Nikiciuk 2010; Saha & Talukdar 2014; El-Nabulsi 2011, 2014, 2017; Davachi & Musielak 2019). Non-standard Lagrangians have not yet been exhaustively studied in physics.

A general non-standard Lagrangian (e.g., Musielak 2009; Cieslinski & Nikiciuk 2010) can be written in the following form

$$L_{ns1}[\dot{x}(t), x(t), t] = \frac{1}{g_1(t)\dot{x}(t) + g_2(t)x(t) + g_3(t)} \ , \tag{2.5}$$

where $g_1(t), g_2(t)$ and $g_3(t)$ are arbitrary, but at least twice differentiable, scalar functions of time t. This Lagrangian can be used to obtain equations of motion for dynamical systems whose coefficients depend only on time, t. However, in the case that the coefficients are functions of the dependent variable, $x(t)$, the non-standard Lagrangian must be of a different form (e.g., Musielak 2009), and written as

$$L_{ns2}[\dot{x}(t), x(t)] = \frac{1}{G_1(x)\dot{x}(t) + G_2(x)x(t) + G_3(x)} \ , \tag{2.6}$$

where $G_1(x), G_2(x)$ and $G_3(x)$ are arbitrary, scalar functions that must be at least twice differentiable. The Lagrangian $L_{ns1}[\dot{x}(t), x(t), t]$ has been extensively studied in the literature (e.g., Musielak 2008, 2009, 2021; Cieslinski & Nikiciuk 2010; and recently Segovia, Vestal, & Musielak 2022). However, studies of $L_{ns2}[\dot{x}(t), x(t)]$ are rather limited (e.g., Musielak 2009; and recently Pham & Musielak 2022). Both Lagrangians are investigated herein and their specific applications to different dynamical systems is presented and discussed in Chapter 3.

The main procedure for finding these two NSLs is to substitute $L_{ns1}[\dot{x}(t), x(t), t]$ and $L_{ns2}[\dot{x}(t), x(t)]$ into the corresponding E-L equations, and then, by comparing the resulting equation to the original (given) equation of motion, the functions are evaluated. The equations of motion must be of specific forms to allow for determining these functions. The Lagrangians $L_{ns1}[\dot{x}(t), x(t), t]$ and $L_{ns2}[\dot{x}(t), x(t)]$ are applicable to the following respective equations of motion

$$\ddot{x} + a(t)\dot{x}^2 + b(t)\dot{x} + c(t)x = 0 \ , \tag{2.7}$$

and

$$\ddot{x} + \alpha(x)\dot{x}^2 + \beta(x)\dot{x} + \gamma(x)x = 0 \ . \tag{2.8}$$

The coefficients $a(t)$, $b(t)$, $c(t)$, $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are at least twice differentiable functions, and they are given by the form of the equation of motion. The functions $g_1(t)$, $g_2(t)$ and $g_3(t)$ are expressed in terms of the coefficients $a(t)$, $b(t)$ and $c(t)$ by substituting $L_{ns1}(\dot{x}, x, t)$ into the E-L equation. However, the functions $G_1(x)$, $G_2(x)$ and $G_3(x)$ are determined by substituting $L_{ns2}(\dot{x}, x)$ into the E-L equation, which allows for these functions to be expressed in terms of the coefficients $\alpha(x)$, $\beta(x)$, and $\gamma(x)$.

The above results show that the form of the derived NSL is very specific for a given equation of motion. For this reason, NSLs are often considered to be the generating functions that allow for finding equations of motion independently from standard Lagrangians. One of the main goals of this dissertation is to gain new insight into the physical meaning of NSLs, which will be done in Chapter 3, where physical examples of NSLs, namely, $L_{ns1}[\dot{x}(t), x(t), t]$ and $L_{ns2}[\dot{x}(t), x(t)]$, are presented for several dynamical systems.

2.3.3 Null Lagrangians

Lagrangians belonging to the family of Lagrangians called trivial, or null, Lagrangians (NLs) have two main characteristics, namely, they identically yield zero from the E-L operator (see Eq. 2.1), and they have corresponding gauge functions. These gauge functions are scalar functions with total derivatives that are equal to NLs. These properties make it such that NLs do not contribute to the equations of motion resulting from the E-L equation as standard or non-standard Lagrangians do when substituted into it. Thus, from a physical point of view, one may consider NLs to be of no interest for any physical applications. It is the main goal of this dissertation to demonstrate that this view is incorrect and that NLs do play important roles in physics; it will futher be shown that these roles are very different than those identified by mathematicians in different fields of mathematics.

Studies of NLs in mathematics date back to the early sixties (Edelen 1962; Eriksen 1962) and have continued throughout the years (e.g., Krupka 1973; Ball et al. 1981; Hojman 1984; Olver 1993; and others). Null Lagrangians are also an active current area of research in mathematics (e.g., Crampin and Saunders 2005; Krupka et al. 2010; Olver 2022; and others). However, NLs have not been used for practical applications in physics until more recent work (e.g., Musielak & Watson 2020a,b; Vestal & Musielak 2021; 2023); the only exception is application of NLs in elasticity, where NLs were identified with the energy density function of materials (e.g., Anderson et al. 1999; Saccomandi & Vitolo 2006).

Specifically, recent physical applications involved restoring the Galilean invariance of Lagrangians in Newtonian dynamics (Musielak & Watson 2020a), and introducing non-dissipative forces to dynamical systems (Musielak & Watson 2020b; Vestal & Musielak 2021), nonlinearities (Vestal & Musielak 2023), and dissipative forces (Segovia et al. 2022); notably, the introduction of forces in these applications was done independently from the original approach used by Newton and others. In the described work, it was also recognized that null Lagrangians can be divided into two families, namely, the so-called standard and non-standard null Lagrangians. The primary differences between these two families of NLs are very similar to those aforementioned for the SLs and NSLs. In this dissertation, both standard and non-standard null Lagrangians are investigated; many different applications of these Lagrangians in physics are presented and discussed in Chapters 4, 5, and 6.

Recent studies of null Lagrangians demonstrated that there is a different condition that is obeyed by NLs and, further, that this condition plays the same role for NLs as the E-L equation plays for SLs and NSLs. According to recent work by Das and Musielak (2022), the condition can be written as

$$\frac{dL_{null}(\dot{x},x)}{dt} = \frac{\partial L_{null}}{\partial t} + \dot{x}\frac{\partial L_{null}}{\partial x} + \ddot{x}\frac{\partial L_{null}}{\partial \dot{x}} = 0 \ , \tag{2.9}$$

and it shows that the substitution of any NL into Eq. (2.6) results in an equation of motion. However, according to Das & Musielak (2022), the resulting equations of motion are limited because their coefficients are required to obey relationships that are different for different equations of motion. It was recently shown (Das & Musielak 2023) that the limitations of this approach can be removed by a significant generalization of the original method and its basic Eq. (2.9); for more details, see Chapter 5.

In the work performed by Das & Musielak (2022, 2023), it was also demonstrated that the inverse of any null Lagrangian generates a non-standard Lagrangian, whose substitution into the E-L equation gives a new equation of motion. This is an interesting result that shows the close relationship between NLs and NSLs. This relationship allows for the construction of new NSLs for all known NLs, and for new equations of motion for dynamical systems to be obtained. The results of Das & Musielak (2022, 2023) will be used in Chapter 5 to relate the derived NLs to NSLs and the resulting equations of motion will be discussed.

2.4 Main Goals of this Dissertation

The primary goal of this dissertation is to communicate the impact and potential of null Lagrangians for research in physics, as well as the work being done. Herein, I aim to show how null and non-standard Lagrangians can be used to characterize systems in physics; the addition of a null Lagrangian to a standard Lagrangian for a given system is shown to be sufficient to convert an undriven, conservative system to one that is driven.

I also present a generalized formalism for directly relating forces and gauges. Forces arise out of gauge terms by way of their corresponding null Lagrangians. This formalism is a new way of introducing forces to systems in physics. Further, nonlinearities can be introduced to a system also through the addition of a null Lagrangian.

Another goal of this dissertation is to present the work that has been done thus far concerning null Lagrangians and gauge functions for systems in physics. As the study of null Lagrangians is relatively new in physics, it is advantageous to discuss the work that has been done. Further, null Lagrangians are presented for special functions of key importance in physics. Finally, it is also shown how null Lagrangians can be written for systems in quantum mechanics.

As much of our physical universe is not yet understood, investigating underlying symmetries, such as the way gauge terms are able to introduce forces and nonlinearities to a system, may shed light on phenomena not as-of-yet understood. Viewing physical phenomena through this new lens may be the key to new connections and compelling discoveries.

CHAPTER 3

Standard and Non-standard Lagrangians for Dynamical Systems

Standard Lagrangians (SLs) have been widely used in multiple areas of physics since being originally introduced by Lagrange in 1788 (see Lagrange 1997), and their specific applications to derive the equations of motion for several well-known dynamical systems will be discussed in this chapter. However, non-standard Lagrangians have been used only recently, and the main reason for this is the fact that NSLs lack distinguishable kinetic and potential energy terms, thus, their physical meaning still remains unclear. Therefore, the emphasis of this chapter will be mainly on NSLs, as SLs are commonly known and presented in most textbooks of classical mechanics and dynamics. One of the goals of this dissertation is to present NSLs for several well-known dynamical systems and also to give new insight into the physical meaning of the NSLs as well as to discuss their potentials as new and promising tools to study dynamical systems (e.g., Goldstein et al. 2002; José & Saletan 2002). The following presentation of SLs and NSLs begins with Newton's law of inertia.

3.1 The Law of Inertia

3.1.1 Standard and Non-standard Lagrangians

Newton's first law, also called the law of inertia, describes the motion of an object with an unchanging velocity. It states that a body at rest will stay at rest and a body in motion at a constant speed will maintain this motion unless acted upon by an outside force (see Goldstein et al. 2002; Jose & Saletan 2002). For the law of inertia, the SL is well known and was originally derived by Lagrange in 1788 (see Lagrange 1997); it can be written as

$$L_s(\dot{x}) = \frac{1}{2}\dot{x}^2. \tag{3.1}$$

Substitution of $L_s(\dot{x})$ into the E-L equation gives the equation of motion, $\ddot{x} = 0$ (e.g., Goldstein et al. 2002; Jose & Saletan 2002). Notably, while the equation of motion is invariant with respect to all transformations that form the Galilean group of the metric (Landau & Lifshitz 1969; Levy-Leblond 1963, 1967), its SL is not; however, it was recently shown that the Galilean invariance of $L_s(\dot{x})$ can be restored by using a null Lagrangian (Musielak & Watson 2020). Further discussion of this problem can be found in Chapter 5.

A general form of NSLs is given by Eq. (2.5) and, after evaluation of its functions $g_1(t), g_2(t)$, and $g_3(t)$ by using the law of inertia, the following NSL is obtained

$$L_{ns}[\dot{x}(t), x(t), t] = \frac{1}{C_1(a_o t + v_o)^2[(a_o t + v_o)\dot{x}(t) - a_o x(t) + C_2]}, \tag{3.2}$$

where C_1 and C_2 are constants of integration, with v_o and a_o specified by the initial conditions for solving the auxiliary differential equation (Musielak 2021). An interesting result is that this Lagrangian gives the law of inertia, which is conservative, despite the fact that it explicitly depends on time. This non-standard Lagrangian was used by Segovia, Vestal, & Musielak (2022) to introduce dissipative forces into the law of inertia and convert it into the second law of dynamics (see also Chapter 5).

Recently, Das & Musielak (2022) considered another NSL for the law of inertia; the form of this Lagrangian was fairly simple:

$$L_{ns}[\dot{x}(t)] = \frac{1}{C\dot{x}(t)}, \tag{3.3}$$

where C is an arbitrary constant. Comparison of Eq. (3.2) to Eq. (3.3) shows the same equation of motion is obtained by the two very different NSLs, with the first one being explicitly time-dependent, and the second one time-independent. Since Newton's law of inertia is the fundamental equation in Galilean relativity, its Galilean invariance must be guaranteed (Landau & Lifshitz 1969). Similarly, the second law of dynamics is also required to be Galilean invariant; however, the latter depends on the form of external force used in the equation

3.1.2 Galilean Invariance

The background space and time of non-relativistic classical mechanics is described by the Galilean metrics $ds_1^2 = dt^2$ and $ds_2^2 = dx^2 + dy^2 + dz^2$, where t is time and x, y and z are Cartesian coordinates associated with an inertial frame of reference (Goldstein et al. 2002; Jose and Salatan 2002). The metrics are invariant with respect to rotations, translations, and boosts, which form the Galilean transformations or the Galilean group of the metric, whose structure is $G = [T(1) \otimes R(3)] \otimes_s [T(3) \otimes B(3)]$; $T(1)$, $R(3)$, $T(3)$ and $B(3)$ are the subgroups of translation in time, rotations in space, translations in space, and boosts, respectively (Levy-Leblond 1963, 1967). The subgroups $T(1)$, $T(3)$ and $B(3)$ are Abelian Lie groups; however, the subgroup $B(3)$ is a non-Abelian Lie group. The direct product is denoted as $\otimes$, and $\otimes_s$ denotes the semi-direct product.

In Newtonian dynamics, the Galilean transformations induce a gauge transformation, which is called the Galilean gauge (Levy-Leblond 1969). The presence of this gauge guarantees that Newton's law of inertia is invariant with respect to the Galilean transformations, and it also shows that its standard Lagrangian is not (Landau & Lifshitz 1969, Levy-Leblond 1969). As shown by Musielak & Watson (2020a), the Galilean invariance of this Lagrangian can be restored through the addition of a null Lagrangian and its gauge function; more details are given in Chapter 5.

To demonstrate the Galilean invariance of the non-standard Lagrangian given by Eq. (3.2), we follow Segovia, Vestal, & Musielak (2022) and start with this Lagrangian in the following form

$$L_{ns}(\dot{x}, x, t) = \frac{1}{C_1 f^2(t)[f(t)\dot{x} - a_o x + C_2]} \ , \tag{3.4}$$

where $f(t) = a_o t + v_o$. After the Galilean transformations, the Lagrangian becomes

$$L'_{ns}[\dot{x}'(t'), x'(t'), t'] = \frac{1}{C'_1 f'^{\,2}(t')[f'(t')\dot{x}'(t) - a'_o x'(t') + C'_2 + v'_o V_0]} \ . \tag{3.5}$$

Galilean invariance of $L_{ns}(\dot{x}, x, t)$ requires that its form is the same as $L'_{ns}[\dot{x}'(t'), x'(t'), t']$. For the original and transformed Lagrangians to be of the same form in the variables $x(t)$ and $x'(t')$, the following conditions must be satisfied: (i) $f'(t') = f(t)$, which requires that $a'_o = a_o$ and $v'_o = v_o$; further, it is also required that $t' = t$, as guaranteed by the Galilean transformation; (ii) $C'_1 = C_1$ is satisfied in all intertial frames; and (iii) $C'_2 + v_o V_0 = C_2$ is valid for all Galilean observers.

Since a_o and v_o are the integration constants for the auxiliary equation (Musielak 2021), and C_1 and C_2 are the constants of integration for the law of inertia, these constants are determined by the initial conditions to be specified for a given physical problem to be solved. However, both the auxiliary equation and the law of inertia are Galilean invariant; thus, the solutions to these equations must be also (Galilean invariant) in all inertial frames. The latter is equivalent to the requirement that the specified initial conditions are also the same for all Galilean observers, which validates the above conditions (i) and (ii). The condition (iii) shows that $C'_2 \neq C_2$ and that the constant C'_2 must be modified by adding another constant $v_o V_0$ to it as compared to C_2. This addition is known in advance by all Galilean observers, who, by their definition, already agreed on the Galilean invariance.

Thus, the non-standard Lagrangian for the law of inertia given by Eq. (3.2) is Galilean invariant, which distinguishes it from the standard Lagrangian, whose original form is not Galilean invariant (Landau & Lifshitz, 1969), and from the non-standard Lagrangian given by Eq. (3.3), which is not Galilean invariant; for further discussion of this non-standard Lagrangian, see Chapter 5.

3.2 Harmonic Oscillators

In classical mechanics, a harmonic oscillator is a system that undergoes simple harmonic motion about an equilibrium point when displaced from that point. The restoring force for this system depends on the spring constant and linearly on the system's displacement from the equilibrium. Thus, the force can be written as

$$F = -kx, \tag{3.6}$$

where x is the displacement of the center of mass of the oscillator from its equilibrium position, F is the restoring force, and k is the spring constant; this is referred to as Hooke's Law (Goldstein et al., 2002; Jose & Saletan, 2002). Then, the resulting equation of motion is

$$\ddot{x} + \omega_o^2 x = 0, \tag{3.7}$$

where $\omega_o^2 = k/m$ is the natural frequency of the oscillator.

The standard Lagrangian for this simple harmonic oscillator has been known since Lagrange's original work (see Lagrange 1997) and is typically written as

$$L_s(\dot{x}, x) = \frac{1}{2}\left(\dot{x}^2 - \omega_o^2 x^2\right). \tag{3.8}$$

Substitution of this Lagrangian into the E-L equation gives Eq. (3.7). However, the non-standard Lagrangian for this system was found by Havas (1957), which is as follows,

$$L_{ns}(\dot{x}, x, t) = \frac{\dot{x}}{\omega_o x} \arctan\left(\frac{\dot{x}}{\omega_o x}\right) - \frac{1}{2} \ln\left(\dot{x}^2 + \omega_o^2 x^2\right) . \tag{3.9}$$

Comparison of $L_s(\dot{x}, x)$ to $L_{ns}(\dot{x}, x, t)$ shows that the form of the latter is more complicated than that of the former. A different form of $L_{ns}(\dot{x}, x, t)$ for the simple harmonic oscillator was derived by Das & Musielak (2022b), who related their non-standard Lagrangian directly to a null Lagrangian and its gauge function (see Chapter 5).

3.3 Bateman Oscillators and the Caldirola-Kanai Lagrangian

Bateman (1931) proposed a model that consists of two uncoupled oscillators, with one oscillator in the model being damped or time-forward while the second one is amplified or time-reversed; this model is now referred to as the Bateman model or the Bateman oscillators (e.g., Vestal & Musielak 2021). The equations of motion for the Bateman model are

$$m\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) = 0 \ , \tag{3.10}$$

and

$$m\ddot{y}(t) - \gamma\dot{y}(t) + ky(t) = 0 \ , \tag{3.11}$$

where $x(t)$ and $y(t)$ are coordinate variables, and $\dot{x}(t)$ and $\dot{y}(t)$ are their derivatives with respect to time t. Further, m is mass, γ represents the damping coefficient, and k is the spring constant. The equations of motion given by Eqs (3.10) and (3.11) describe damped and amplified oscillators, respectively. The equations are uncoupled; however, they are related to each other by the transformation $[x(t), y(t), \gamma] \to [y(t), x(t), -\gamma]$, which allows for the replacement of Eq. (3.10) by Eq. (3.11) and vice versa. Notably, the general solutions of the equations of motion for the Bateman model are well-known and given in terms of elementary functions (e.g., Murphy 2011, Razavy 2017).

As originally shown by Bateman (1931), the equations of motion given by Eqs (3.10) and (3.11) can be derived from the following Lagrangian, which is known in the literature as the Bateman Lagrangian

$$L_B[\dot{x}(t), \dot{y}(t), x(t), y(t)] = m\dot{x}(t)\dot{y}(t) + \frac{\gamma}{2}\left[x(t)\dot{y}(t) - \dot{x}(t)y(t)\right]$$
$$-kx(t)y(t) \ . \tag{3.12}$$

By substituting this Lagrangian into the E-L equations for $y(t)$ and $x(t)$, the resulting equations of motion for the damped and amplified oscillators are obtained (e.g., Vujanovic and Jones 1989). The Bateman Lagrangian has also been used to quantize the damped harmonic oscillator (Weiss 2008, Razavy 2017; see also Pais & Uhlenbeck 1950, Feshbach & Tikochinsky 1977, Deguchi et al. 2019, Bagarello et al. 2019, 2020).

Let us follow Vestal & Musielak (2021) and define $b = \pm\gamma/m$ and $c = k/m = \omega_o^2$, where ω_o is the characteristic frequency of the oscillators. Then, we write Eqs (3.10) and (3.11) as one equation of motion

$$\ddot{x}(t) + b\dot{x}(t) + \omega_o^2 x(t) = 0 \ , \tag{3.13}$$

with the understanding that the damped and amplified oscillators require $b > 0$ and $b < 0$, respectively, and that the variable $x(t)$ describes either damped or amplified oscillator. Let $\hat{D} = d^2/dt^2 + bd/dt + \omega_o^2$ be a linear operator; Eq. (3.13) can then be written in the compact form $\hat{D}x(t) = 0$.

The first-derivative term in Eq. (3.13) can be removed by using the standard transformation of the dependent variable (Kahn 1990). The transformation is

$$x(t) = x_1(t)e^{-bt/2} \ , \tag{3.14}$$

where $x_1(t)$ is the transformed dependent variable, and it gives

$$\ddot{x}_1(t) + \left(\omega_o^2 - \frac{1}{4}b^2\right) x_1(t) = 0 \ . \tag{3.15}$$

Despite the fact that the first derivative term is removed, the coefficient b is still present in the transformed equation of motion. However, if $b = 0$, then $x_1(t) = x(t)$ and Eq. (3.15) becomes the equation of motion for a undamped harmonic oscillator (Goldstein et al., 2002 José & Saletan, 2002).

The standard Lagrangian for this equation is

$$L_s[\dot{x}_1(t), x_1(t)] = \frac{1}{2}\left[(\dot{x}_1(t))^2 - \left(\omega_o^2 - \frac{1}{4}b^2\right) x_1^2(t)\right] \ , \tag{3.16}$$

and its substitution into the E-L equation gives Eq. (3.15).

The derived equation of motion and its standard Lagrangians are expressed in terms of the dynamical variable $x_1(t)$. Let us now use Eq. (3.14) to make the inverse transformation and convert the variable $x_1(t)$ into $x(t)$ in $L_s[\dot{x}_1(t), x_1(t)]$. The resulting Lagrangian is

$$L_s[\dot{x}(t), x(t), t] = L_{CK}[\dot{x}(t), x(t), t] \ , \tag{3.17}$$

where

$$L_{CK}[\dot{x}(t), x(t), t] = \frac{1}{2}\left[(\dot{x}(t))^2 - \omega_o^2 x^2(t)\right] e^{bt} \ , \tag{3.18}$$

is the Caldirola-Kanai (CK) Lagrangian (Caldirola 1941, Kanai 1948), which is derived here independently; comparison of the CK and Bateman Lagrangian (Eqs (3.18) and (3.12)) shows significant differences in their forms and physical meaning.

The CK Lagrangian, $L_{CK}[\dot{x}(t), x(t), t]$, when substituted into the E-L equation, yields $[\hat{D}x(t)]e^{bt} = 0$, which is consistent with all Helmholtz conditions that are valid for any system of ordinary differential equations (Helmholtz 1887, Vujanovic & Jones 1989). However, with $e^{bt} \neq 0$, the resulting $\hat{D}x(t) = 0$ does obey the first and second

Helmholtz conditions (see Eqs (2.2) and (2.3)), but fails to satisfy the third condition (see Eq. (2.4)). This shows that, after the division by e^{bt}, the equation of motion fails to satisfy the third Helmholtz condition (Musielak, Vestal, et al. 2020).

The role of the CK Lagrangian in deriving the equation of motion for the Bateman oscillators has been questioned in the literature (Ray 1979, Segovia-Chavez 2018) and it was concluded that the CK Lagrangian does not describe the Bateman oscillators but instead a different oscillatory system in which the mass is increasing ($b > 0$) or decreasing ($b < 0$) exponentially in time. However, as pointed out recently by Torres del Castillo (2019), the previous work has some conceptual errors that led to incorrect conclusions.

The results presented by Vestal & Musielak (2021) and described in this dissertation show that the total energy and the linear momentum decrease (increase) in time for the damped (amplified) Bateman oscillators, which is consistent with the physical picture of these dynamical systems. The increase (decrease) of the canonical momentum in time does not contradict this picture, but instead it guarantees that the CK Lagrangian can be used to derive the equations of motion for the Bateman oscillators. These results and conclusions are consistent with those previously obtained by and described by Torres del Castillo (2019).

Having obtained and discussed the standard (CK) Lagrangian for the Bateman oscillators, the non-standard Lagrangian for these dynamical systems will now be presented. As already shown in Section 3.2, the non-standard Lagrangian for an undamped harmonic oscillator was found by Havas (1957) (see Eq. 3.9). Let us now modify Havas' non-standard Lagrangian so that it accounts for damping. This modification requires that we multiply the Lagrangian by the exponential term e^{bt}, as was done for the CK Lagrangian, and the result is

$$L_{ns}(\dot{x}, x, t) = \left[\frac{\dot{x}}{\omega_o x} \arctan\left(\frac{\dot{x}}{\omega_o x} \right) - \frac{1}{2} \ln\left(\dot{x}^2 + \omega_o^2 x^2 \right) \right] e^{bt}. \tag{3.19}$$

Comparison of this non-standard Lagrangian to the standard (CK) Lagrangian shows that the former is more complicated than the latter. Further, the fact that there are also different forms of non-standard Lagrangians for the Bateman oscillators has been recently demonstrated by Das & Musielak (2023).

3.4 Special Functions of Mathematical Physics

Special functions (SFs) are of key interest in mathematical physics as they are solutions of second-order ordinary differential equations (ODEs). In this way, they play important roles in both classical and quantum physics. Typically, ODEs whose solutions are special functions are obtained by using the method of separation of variables in hyperbolic, parabolic, and elliptic partial differential equations (PDEs) (e.g., Cantrell, 2000; Bayin, 2006; Murphy, 2011). Another (lesser known) method is based on Lie groups, whose irreducible representations (irreps) are used to find the SFs and their corresponding ODEs (e.g., Miller 1968, Nikiforov & Uvarov 1988, Cantrell 2000, Mathai & Haubold 2008). The Lagrangian formalism for the ODEs with SFs solutions was established by Musielak et al. (2020). In this section, some key results pertaining to standard and non-standard Lagrangians from this work will be presented.

To introduce ODEs with SF solutions, let us first change the notation used in the previous sections in this chapter, where dynamical systems were considered; x was used as their dependent variable, and t was the independent variable. Since SFs are solutions to ODEs describing different physical and mathematical systems, which may or may not be dynamical ones, in the notation used in this section, the dependent variable is y and x is the independent variable. One of the justifications for this choice is the fact that this notation is commonly used in applied mathematics, physics, and engineering. In the following, the presented standard and non-standard Lagrangians were originally obtained by Musielak et al. (2020), and they are presented here in preparation for a more detailed discussion of null Lagrangians and their gauge functions derived for the ODEs with SF solutions by Dange, Vestal, & Musielak (2021) - see Chapter 5.

Let $\hat{D} = d^2/dx^2 + B(x)d/dx + C(x)$ be a linear operator with $B(x)$ and $C(x)$ being ordinary (with the maps $B: \mathcal{R} \to \mathcal{R}$ and $C: \mathcal{R} \to \mathcal{R}$, with $\mathcal{R}$ denoting the real numbers) and smooth functions with at least two continuous derivatives ($\mathcal{C}^2$) defined either over a restricted interval (a, b) or an infinite interval $(-\infty, \infty)$, depending on the form of the ODE with SF solutions. Now, let $\hat{D}y(x) = 0$ be a linear second-order ODE with non-constant coefficients. The functions $B(x)$ and $C(x)$ can then be selected such that the resulting equations represent all the ODEs whose solutions are SFs. In the following, the general forms of constructed standard and non-standard Lagrangians are presented and then used to find such Lagrangians for the Bessel, Legendre, and Hermite equations.

The constructed standard Lagrangians, denoted here as L_s, are of the form:

$$L_s[y'(x), y(x), x] = G_s[y'(x), y(x), x] \; E_s(x) \; , \tag{3.20}$$

where

$$G_s[y'(x), y(x), x] = \frac{1}{2} \left[(y'(x))^2 - C(x)y^2(x) \right] \; , \tag{3.21}$$

and $E_s(x) = \exp\left[\int^x B(\tilde{x})d\tilde{x}\right]$. Since $E_s(x)$ is only a function of one independent variable, x, the lower limit must be an arbitrary constant, which can be omitted because such constant has no effect on the Lagrangian formulation. Note that in a special case of $B(x)$ = constant, $L_s[y'(x), y(x), x]$ becomes the Caldirola–Kanai Lagrangian (see Section 3.3, and also Vestal & Musielak 2021).

To construct non-standard Lagrangians for the ODEs with the SFs solutions, Dange, Vestal, & Musielak (2021) considered the form of such Lagrangian as given by Eq. (2.5) and wrote it as $L_{ns}[y'(x), y(x), x] = 1/[f(x)y'(x) + g(x)y(x)]$, where the functions $f(x)$ and $g(x)$ are to be determined for a given ODE. In general, the non-standard Lagrangian is

$$L_{ns}[y'(x), y(x), x] = H_{ns}[y'(x), y(x), x] \ E_{ns}(x) \ , \tag{3.22}$$

where

$$H_{ns}[y'(x), y(x), x] = \frac{1}{[y'(x)\bar{v}(x) - y(x)\bar{v}'(x)] \ \bar{v}^2(x)} \tag{3.23}$$

and $E_{ns}(x) = \exp\left[-2 \int^x B(\tilde{x}) d\tilde{x}\right]$, with the necessary auxiliary condition $\hat{D}\bar{v}(x) = 0$.

Let us now apply the above results to the Bessel, Legendre, and Hermite equations, for which both standard and non-standard Lagrangians are constructed. The standard Lagrangians are constructed using Eqs (3.20) and (3.21), and for the non-standard Lagrangians Eqs (3.22) and (3.23) are used.

Bessel equation

Let us write the Bessel equation in the following general form

$$y''(x) + \frac{\alpha}{x} y'(x) + \beta \left(1 + \gamma \frac{\mu^2}{x^2}\right) y(x) = 0 \ , \tag{3.24}$$

where $B(x) = \alpha/x$ and $C(x) = \beta(1 + \gamma\mu^2/x^2)$. In addition, α, β, γ, and μ are constants, and their specific values determine the four different types of Bessel equations (see Table 3.1).

Then, the standard and non-standard Lagrangians for Eq. (3.24) are

$$L_s[y'(x), y(x), x] = \frac{1}{2} \left[(y'(x))^2 - \beta \left(1 + \gamma \frac{\mu^2}{x^2}\right) y^2(x) \right] x^\alpha \ , \tag{3.25}$$

and

$$L_{ns}[y'(x), y(x), x] = H_{ns}[y'(x), y(x), x] \; x^{-2\alpha} \; , \tag{3.26}$$

where $H_{ns}[y'(x), y(x), x]$ is given by Eq. (3.23),

The auxiliary condition that must supplement $L_{ns}[y'(x), y(x), x]$ is given by

$$\bar{v}''(x) + \frac{\alpha}{x}\bar{v}'(x) = -\beta\left(1 + \gamma\frac{\mu^2}{x^2}\right)\bar{v}(x) \; , \tag{3.27}$$

and this condition is required in order to derive the original Bessel equation given by Eq. (3.24) from the E-L equation.

Legendre Equations

There are the regular and associated Legendre equations, and the latter can be written as

$$y''(x) - \frac{2x}{(1-x^2)}y'(x) + \left[\frac{l(l+1)}{(1-x^2)} - \frac{m^2}{(1-x^2)^2}\right]y(x) = 0 \; , \tag{3.28}$$

where l and m are constants, and when $m = 0$ the above equation becomes the regular Legendre equation. Then, the standard and non-standard Lagrangians are

$$L_s[y'(x), y(x), x] = \frac{1}{2}\left[y'(x)\right]^2(1-x^2)$$

$$- \left[\frac{l(l+1)}{(1-x^2)} - \frac{m^2}{(1-x^2)^2}\right]y^2(x)(1-x^2) \; , \tag{3.29}$$

$$L_{ns}[y'(x), y(x), x] = H_{ns}[y'(x), y(x), x] \; (1-x^2)^{-2} \; , \tag{3.30}$$

where $H_{ns}[y'(x), y(x), x]$ is given by Eq. (3.24).

Hermite Equation

The Hermite equation can be written as

$$y''(x) - xy'(x) + ny(x) = 0 \; , \tag{3.31}$$

where n is any integer. Comparing this equation to $\hat{D}y(x) = 0$, we find $B(x) = -x$ and $C(x) = n$. The range of validity of the Hermite equation is $x \;\epsilon\; (0, \infty)$.

The explicit forms of the standard and non-standard Lagrangians for the Hermite equation are

$$L_s[y'(x), y(x), x] = \frac{1}{2}\left[(y'(x))^2 - ny^2(x)\right] e^{-x^2/2} \ , \tag{3.32}$$

$$L_{ns}[y'(x), y(x), x] = H_{ns}[y'(x), y(x), x] e^{x^2} \ , \tag{3.33}$$

where $H_{ns}[y'(x), y(x), x]$ is given by Eq. (3.24). The auxiliary condition given by

$$\bar{v}''(x) + x\bar{v}'(x) = -n\bar{v}(x) \ , \tag{3.34}$$

must also be used to derive the original equation from the non-standard Lagrangian.

The above results presented here by following Musielak et al. (2020) have been significantly extended by Dange, Vestal, & Musielak (2021), who constructed null Lagrangians and their gauge functions for ODEs with SF solutions (see Chapter 4).

3.5 Biological Applications

Another compelling application of Lagrangians has been to biological systems. Lagrangians were first applied in the field of theoretical biology by Kerner (1964). Kerner noted that many systems in biology, such as population kinetics in ecological theory, are given in terms of first-order ODEs. Later, Paine (1982) studied Lagrangians for similar sets of ODEs following the original work of Helmholtz (1887). The first specific applications of the Lagrangian formalism to population dynamics were done by Trubatch & Franco (1974), who obtained in an ad hoc manner Lagrangians for the population dynamics models such as the Lotka-Volterra (Lotka, 1925; Volterra, 1926), Verhulst (1838), Gompertz (1825), and Host-Parasite (Collins et al., 1956) models. The Lagrangians found for the models were formally derived by Nucci & Tamizhmani (2012), who used the method based on the Jacobi Last Multiplier (Nucci & Leach 2007). The derived Lagrangians were non-standard and therefore treated as generating functions whose substitution into the E-L equation gave the evolution equations for the models.

Among a large variety of biological systems, population dynamics plays a special role as it is the key to understanding the relative importance of the competition for resources and predation in complex communities, and for preserving biodiversity (Turchin, 2003a,b; Oro, 2013). Population dynamics models that describe interacting species are typically expressed by ODEs, which are first-order, coupled, damped, and nonlinear (Turchin, 2003a,b). Despite the presence of damping and nonlinearities in such models, no clear demonstration of the onset of chaos has yet been shown (e.g., Rai & Upadhyay, 2004). However, some studies have suggested that insect population dynamics can undergo transitions between stable and chaotic phases for models near a transition point between order and chaos (e.g., Figueroa et al., 2020).

In recent work (Pham & Musielak, 2023a), the standard Lagrangians were constructed for the above listed population dynamics models with one additional model included; this is the SIR model (Kermack & McKendrick, 1927) and it describes the spread of a disease in a population. In the constructed standard Lagrangians, kinetic energy-like terms and potential energy-like terms were identified and their roles in population dynamics were discussed. It was also suggested that the identification of these energy-like terms may become an efficient way of comparing biological models. Further, it was found that force-like terms appear for some of these models. For the models of population dynamics for interacting species, an oscillator-like behavior, with respect to a given system's equilibrium, was observed. In the following, a brief description of the results obtained by Pham & Musielak (2023a) is presented.

Typical equations of motion of population dynamics models can be written in the following form

$$\ddot{x} + \alpha(x)\dot{x}^2 + \gamma(x)x = F(x, \dot{x}), \tag{3.35}$$

where $x(t)$ represents the population of species and the force-like term is given by

$$F(x, \dot{x}) = C_0 - \beta(x)\dot{x}, \tag{3.36}$$

with C_0 being a constant. Note that $F(x, \dot{x})$ has the dissipative term $\beta(x)\dot{x}$, which is by itself a null Lagrangian (see Chapter 4).

According to the method developed by Pham & Musielak (2023a), the constructed standard Lagrangian can be written as

$$L(\dot{x}, x) = \frac{1}{2}\dot{x}^2 e^{2I_\alpha(x)} - \int^x \tilde{x}\gamma(\tilde{x})e^{2I_\alpha(\tilde{x})}d\tilde{x} \ , \tag{3.37}$$

where

$$I_\alpha(x) = \int^x \alpha(\tilde{x})d\tilde{x} \ , \tag{3.38}$$

and $\hat{EL}[L(\dot{x}, x)] = F(\dot{x}, x)e^{2I_\alpha(x)}$ or more explicitly

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F(\dot{x}, x)e^{2I_\alpha(x)} \ . \tag{3.39}$$

The presence of the term $F(\dot{x}, x)e^{2I_\alpha(x)}$ is justified by the fact that this term does not arise from any potential (e.g., Goldstein et al., 2002). The presented method was used by Pham & Musielak (2023a) to construct standard Lagrangians for the following population dynamics systems: the Lotka-Volterra, Verhulst, Gompertz, Host-Parasite, and SIR models.

The above work to construct the standard Lagrangians for the population dynamics models was followed by Pham & Musielak (2023b), who developed another method to construct non-standard Lagrangians. The method requires that the equations of motion for the population dynamics models (see Eq. 3.35) is written as

$$\ddot{x} + \alpha(x)\dot{x}^2 = F_{dis}(\dot{x}, x) \ , \tag{3.40}$$

where

$$F_{dis}(\dot{x}, x) = F(x) - \beta(x)\dot{x} - \gamma(x)x \ , \tag{3.41}$$

becomes a dissipative force because of its dependence on $\dot{x}(t)$. Then, the required E-L equation can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F_{dis}(\dot{x}, x)e^{2I_\alpha(x)} \ , \tag{3.42}$$

where the force on the RHS of this equation is known as the Rayleigh force (Goldstein et al. 2002). The form of the non-standard Lagrangians that must be used to obtain the equation of motion (see Eq. 3.40) is given by Eq. (2.6), and after evaluating the arbitrary functions in this equation, the final Lagrangian becomes

$$L_{ns}(\dot{x}, x) = \frac{1}{\dot{x}e^{I_\alpha(x)} + C_o} \ , \tag{3.43}$$

where C_o is an arbitrary constant.

As demonstrated by Pham & Musielak (2023b), the obtained non-standard Lagrangian has no restrictions or limitations and it exists for any differentiable coefficient $\alpha(x)$ regardless of the forms of $\beta(x)$ and $\gamma(x)$. Therefore, $L_{ns}(\dot{x}, x)$ was used to derive non-standard Lagrangians for the Lotka-Volterra, Verhulst, Gompertz, Host-Parasite, and SIR population dynamics models. An interesting result obtained by Pham & Musielak (2023b) is that the derived non-standard Lagrangians are directly related to null Lagrangians and their gauge function; thus, the first null Lagrangians and gauge functions for the considered models were also derived.

3.6 Summary

While standard Lagrangians have been utilized at length for their ability to efficiently and effectively describe physical systems, non-standard Lagrangians, contrastingly, have not. In this chapter, some standard and non-standard Lagrangians of importance in physics were introduced, and the differences and similarities between these two families of Lagrangians were highlighted. Notably, non-standard Lagrangians do not have distinguishable kinetic and potential energy terms; this attribute differs from their standard Lagrangian counterparts, and in this chapter I discussed how also the non-standard Lagrangian form of the law of inertia is Galilean invariant, which is not true for the standard Lagrangian formulation. Non-standard Lagrangians for harmonic oscillators were then discussed, along with work in this area relevant to non-standard Lagrangians. Our work investigating the Bateman model and how it relates to the CK Lagrangian was also discussed. Special functions of key importance for systems in physics are presented as they will be further discussed in Chapter 4. Lastly, a brief history of the application of Lagrangians to the field of theoretical biology is given; special attention is given to recent work in which standard and non-standard Lagrangians are used for models of population dynamics.

The law of inertia was presented along with its standard Lagrangian form and an equivalent non-standard Lagrangian form. Another system of key interest in physics, the harmonic oscillator, was discussed; a dissipative system of oscillators was also introduced. Some applications of the Bessel, Hermite, and Legendre equations were given. Lastly, biological applications of standard and non-standard Lagrangians were described.

Table 3.1. Values of the constants α, β, γ, and μ in Equation (3.24) corresponding to the four types of Bessel equations.

Bessel Equations	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\mu}$
Regular	1	1	-1	real or integer
Modified	1	-1	1	real or integer
Spherical	2	1	-1	$\mu^2 = l(l+1)$
Modified spherical	2	-1	1	$\mu^2 = l(l+1)$

CHAPTER 4

Standard Null Lagrangians and their Novel Roles in Dynamics

In this chapter, my research involving null Lagrangians is introduced. I walk through the steps for introducing novel gauge functions (GFs) from null Lagrangians to non-relativistic classical mechanics, and how they are used to define forces. The results presented are from multiple projects investigating null Lagrangians and gauges in which I have had a significant role; they show that gauge functions directly affect the energy function and that they allow for the conversion of an undriven physical system into a driven one, which is one of the main focuses of this dissertation. This is a novel phenomenon in dynamics that resembles the role of gauges in quantum field theory. The discussion contained in this chapter is limited to standard null Lagrangians; the chapter to follow will address non-standard null Lagrangians and extend the formalism developed herein.

The layout is as follows: In 4.1, I introduce two methods to construct standard null Lagrangians (SNLs) and I comment on why null Lagrangians are so compelling for research in physics. The connection between NLs and forces is then explored in section 4.2. An application of null Lagrangians to oscillators is presented in section 4.3; gauge functions corresponding to NLs are found and forces are introduced for the harmonic oscillator, demonstrating how the formula may be applied to a simple system and allowing for the conversion of this conservative, undriven system to a driven one. In section 4.4, it is shown also how gauge functions can be used to convert the undriven Bateman oscillators into driven ones; this result is important and novel in that the formalism is extended here to dissipative systems. In section 4.5, a compelling direct link between gauge functions and forces is presented, and I illustrate the connection between these as well as to null Lagrangians. Then, in section 4.6, a new and promising area of NL research is presented. The formalism developed in section 4.5 (general NLs) is used and a new way and is shown to reproduce nonlinearities, extending the list of applications of this method beyond the introduction of forces. This is followed in section 4.7 by a discussion of null Lagrangians for some ODEs with special function (SF) solutions that are of key interest in physics. This project further illustrates the vast range of applications of NLs and GFs by this time focusing on SFs. A short summary is presented at the close of the chapter.

4.1 Methods to Construct Standard Null Lagrangians

Multiple methods to construct NLs have been described in the literature (e.g., Krupka et al., 2010; Olver, 2022; Musielak & Watson, 2020a,b; Vestal & Musielak, 2021; 2023). In this section, I discuss two methods of constructing null Lagrangians and their corresponding gauges. The first method restricts the orders of the dependent and independent variables, yielding uncomplicated NL terms that can be directly added to the Lagrangian of a given system, and its development was presented in several papers (e.g., Musielak & Vestal et al., 2020; Vestal & Musielak 2021; Dange, Vestal, & Musielak 2021). The second method was introduced more recently by Vestal & Musielak (2023). This alternate method, which successfully reproduces NLs, gauge terms, and forces that match those previously published, was developed with the aim of creating a generalized framework for finding all NLs and force terms for a given system.

4.1.1 First Method

In this first method, NLs and their gauge functions are found for second-order ordinary differential equations (ODEs) in one dimension. The constructed NLs are of lower or comparable orders to the standard Lagrangian for a given equation and depend on arbitrary constant coefficients, which are replaced here with arbitrary functions of the independent variable. This is done for a non-dissipative oscillator system, and the NLs developed are further physically investigated in the context of this system in Musielak & Vestal et al. (2020).

For the simple oscillators, the independent variable t is time and the dependent variable x(t) is a displacement. Let $\hat{D} = d^2/dt^2 + c$ be an linear differential operator, with c being a constant whose value may change from one dynamical system to another, and let Q be a set of all ODEs of the form $\hat{D}x(t) = 0$; depending on the physical meaning of x(t) and c, the ODEs of Q may describe different linear oscillators, including linear pendulums. General solutions of these ODEs are well-known and can be written as $x(t) = c_1 x_1(t) + c_2 x_2(t)$, where c_1 and c_2 are integration constants, and $x_1(t)$ and $x_2(t)$ are the solutions given in terms of the elementary functions (Murphy 2011; Teschl 2012). The Lagrangian formalism is established for the ODEs of Q.

First, let $L_m[\dot{x}(t), x(t)]$ be a mixed Lagrangian of dependent and independent variables given by

$$L_m[\dot{x}(t), x(t), t] = C_1 \dot{x}(t) x(t) + C_2 \dot{x}(t) t + C_3 x(t) t \ , \qquad (4.1)$$

and $L_f[\dot{x}(t), x(t)]$ be a Lagrangian of the single dependent variable written as

$$L_f[\dot{x}(t), x(t)] = C_4 \dot{x}(t) + C_5 x(t) + C_6 \ , \qquad (4.2)$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants.

We will use the standard Lagrangian

$$L_s[\dot{x}(t), x(t)] = \frac{1}{2}\left[\alpha\left(\dot{x}(t)\right)^2 + \beta x^2(t)\right] \ , \tag{4.3}$$

where the coefficients α and β are either constants or functions of time.

Because $x(t)$ is a displacement of harmonic oscillators and t is time, the constants must have different physical dimensions for the dimensions of $L_m[\dot{x}(t), x(t), t]$ and $L_f[\dot{x}(t), x(t)]$ to be consistent with those of $L_s[\dot{x}(t), x(t)]$.

We define $\hat{EL}$ to be the E-L equation operator and take $\hat{EL}(L_m + L_f) = 0$. $L_n[\dot{x}(t), x(t), t] = L_m[\dot{x}(t), x(t), t] + L_f[\dot{x}(t), x(t)]$ can then become a null Lagrangian, which requires $C_3 = 0$ and $C_5 = C_2$. Then, the null Lagrangian can be written as

$$L_n[\dot{x}(t), x(t), t] = \sum_{i=1}^{4} L_{ni}[\dot{x}(t), x(t), t] \ , \tag{4.4}$$

where $i = 1$, 2, 3 and 4, and the partial NLs are given by

$$L_{n1}[\dot{x}(t), x(t)] = C_1 \dot{x}(t) x(t), \tag{4.5}$$

$$L_{n2}[\dot{x}(t), x(t), t] = C_2[\dot{x}(t)t + x(t)], \tag{4.6}$$

$$L_{n3}[\dot{x}(t)] = C_4 \dot{x}(t) \tag{4.7}$$

and

$$L_{n4} = C_6 \tag{4.8}$$

Of these, $L_{n2}[\dot{x}(t), x(t), t]$ is the only partial null Lagrangian that depends explicitly on t. Note that these partial null Lagrangians are constructed to lowest orders of the dynamic variable $x(t)$.

The term *partial null Lagrangian* is used here to refer to individual null Lagrangians that are added together to form the null Lagrangian of interest. Each of these terms alone (equations 4.4-4.8) would also be a null Lagrangian. Similarly, each partial null Lagrangian has a corresponding partial gauge function.

We may write the gauge function, $\Phi_p(t)$, for L_n by using the definition from Chapter 2,

$$\Phi_p(t) = \sum_{i=1}^{4} \phi_{pi}(t) \ , \tag{4.9}$$

where the partial gauge functions $\phi_{pi}(t)$ correspond the partial null Lagrangians $L_{ni}[\dot{x}(t), x(t)]$, and they are defined as $\phi_{p1}(t) = C_1 x^2(t)/2$, $\phi_{p2}(t) = C_2 x(t)t$, $\phi_{p3}(t) = C_4 x(t)$ and $\phi_{p4}(t) = C_6 t$.

We consider the ODEs of $\mathcal{Q}$ and write them in their explicit form

$$\ddot{x}(t) + cx(t) = 0 \ , \tag{4.10}$$

where c may be any real number. Let us define the following primary Lagrangian

$$L_p[\dot{x}(t), x(t), t] = L_{ps}[\dot{x}(t), x(t)] + L_{pn}[\dot{x}(t), x(t), t] \ , \tag{4.11}$$

where the primary standard Lagrangian (with $\alpha = 1$ and $\beta = -c$ in Eq. 4.3) is given by

$$L_{ps}[\dot{x}(t), x(t)] = \frac{1}{2}\left[(\dot{x}(t))^2 - cx^2(t)\right] \ , \tag{4.12}$$

and the primary null Lagrangian $L_{pn}[\dot{x}(t), x(t)]$ is equal to $L_n[\dot{x}(t), x(t)]$ (see Eq. 4.4) with the same partial NLs. In addition, the primary gauge function $\Phi_p(t)$ is given by Eq. (4.9) with the same partial gauge functions.

The above results can be generalized by writing the Lagrangian given by Eq. (4.3) in the following form

$$L_s[\dot{x}(t), x(t)] = \frac{1}{2}\left[\alpha(t)\left(\dot{x}(t)\right)^2 + \beta(t)x^2(t)\right] \ , \tag{4.13}$$

where $\alpha(t)$ and $\beta(t)$ are continuous and differentiable functions. Substituting this Lagrangian into the E-L equation, we find $\alpha(t) = C_o$ and $\beta(t) = -C_o c$, where C_o is an intergration constant. Then, the general standard Lagrangian can be written as

$$L_{gs}[\dot{x}(t), x(t)] = \frac{1}{2} C_o \left[(\dot{x}(t))^2 - c x^2(t) \right] \ . \tag{4.14}$$

This Lagrangian can be reduced to the primary standard Lagrangian if $C_o = 1$ and it can also be used to define the following general Lagrangian

$$L_g[\dot{x}(t), x(t), t] = L_{gs}[\dot{x}(t), x(t), t] + L_{gn}[\dot{x}(t), x(t), t] \ , \tag{4.15}$$

where the general null Lagrangian is

$$L_{gn}[\dot{x}(t), x(t), t] = \sum_{i=1}^{4} L_{gni}[\dot{x}(t), x(t), t] \ , \tag{4.16}$$

with $\hat{EL}(L_{gn}) = 0$ and $L_{gni}[\dot{x}(t), x(t), t]$ being its partial components. To determine the partial null Lagrangians, we generalize the primary gauge functions $\phi_{pi}(t)$ given below Eq. (4.9) by replacing their constant coefficients by functions of the independent variable t. Denoting the general gauge functions as $\phi_{gi}(t)$, we obtain

$$\phi_{g1}(t) = \frac{1}{2} f_1(t) x^2(t) \ , \tag{4.17}$$

$$\phi_{g2}(t) = f_2(t) x(t) t \ , \tag{4.18}$$

$$\phi_{g3}(t) = f_4(t) x(t) \ , \tag{4.19}$$

and

$$\phi_{g4}(t) = f_6(t) t \ , \tag{4.20}$$

where $f_1(t)$, $f_2(t)$, $f_4(t)$ and $f_6(t)$ are continuous and differentiable functions to be determined.

Then, we take the total derivatives of these partial gauge functions and obtain the following partial Lagrangians

$$L_{gn1}[\dot{x}(t), x(t), t] = \left[f_1(t)\dot{x}(t) + \frac{1}{2}\dot{f}_1(t)x(t) \right] x(t) \ , \tag{4.21}$$

$$L_{gn2}[\dot{x}(t), x(t), t] = \left[\left(f_2(t)\dot{x}(t) + \dot{f}_2(t)x(t) \right) t + f_2(t)x \right] \ , \tag{4.22}$$

$$L_{gn3}[\dot{x}(t), x(t), t] = \left[f_4(t)\dot{x}(t) + \dot{f}_4(t)x(t) \right] \ , \tag{4.23}$$

and

$$L_{gn4}[\dot{x}(t), x(t), t] = \left[\dot{f}_6(t)t + f_6(t) \right] \ , \tag{4.24}$$

which can be added together to obtain the general null Lagrangian (see Eq. 4.16). This Lagrangian depends on four functions that must be continuous and differentiable but are otherwise arbitrary. Specification of the initial conditions for physical problems would set up constraints on these functions, however, in this section the functions are kept arbitrary for reasons explained in section 4. The general null Lagrangian reduces to the primary null Lagrangian when $f_1(t) = C_1,\ f_2(t) = C_2,\ f_4(t) = C_4$ and f_6

We derived the primary and general SLs and NLs for the ODEs of Q. Most obtained SLs are already known and they are generated as a byproduct of our procedure of deriving the NLs, which are new for the considered equations. For each null Lagrangian, we found its corresponding gauge function. The general Lagrangians depend on four functions that must be continuous and differentiable, and must satisfy initial conditions of a specific physical problem. If the functions are assumed to be constants, the primary NLs are obtained. Since the functions are arbitrary, many different NLs can be obtained by choosing different forms of these functions.

4.1.2 Second Method

The second method, which is capable of reproducing the NLs found above, will now be introduced. Further results from this method will be presented later in the Chapter, in Table 4.1. In investigating this generalized approach to building NLs staring from their gauges (Φ), rather than starting from a NL and finding its corresponding gauge after, a clear link between the form of a given NL and the form of the force-like term it produces was established.

This dissertation draws a clear line from the gauge term to its corresponding NL, and then to the resulting energy function and finally to the resulting force-like term. This reveals a fundamental connection underlying dynamical systems and sheds more light on the impact and significance of NLs for physical systems. Further, it was shown that this formalism can reproduce nonlinear terms by Vestal & Musielak (2023), which is discussed later in the Chapter.

If L_n is a NL and $\hat{EL}$ is the Euler-Lagrange operator, then $\hat{EL}(L_n) = 0$, as required by the definition of the NLs. Such NLs may depend on both the dependent and independent variables, so they can be written as $L_n(\dot{x}, x, t)$. These NLs can be added to the standard Lagragian, $L_s(\dot{x}, x, t)$, without having any effect on the resulting equation of motion. Thus, the total Lagrangian, L_{tot}, can be written as

$$L_{tot}(\dot{x}, x, t) = L_s(\dot{x}, x, t) + L_n(\dot{x}, x, t) \ , \tag{4.25}$$

where $L_n(\dot{x}, x, t)$ is the NL that can be obtained by calculating the total derivative of any scalar and differentiable function (Olver 1993; Olver & Sivaloganathan 1989; Crampin & Saunders 2005; Vitolo 1999), which is called the gauge function and denoted as $\Phi(x, t)$, with $x = x(t)$. Thus, the general NL can be written using Eq. (2.9). Substitution of this NL into the E-L equation shows that the gauge function can be of any form as long as it has the properties specified above.

Despite the fact that the presence of NLs does not change the form of the resulting equation of motion, it has been shown that the NLs may be used to restore Galilean invariance of standard Lagrangians (eg. Levy-Leblond 1969; Musielak & Watson 2020; Segovia, Vestal, & Musielak 2022), and to independently introduce forces to classical mechanics by some, but not all, NLs by Musielak & Vestal et al. (2020), and by Vestal & Musielak (2023). Therefore, it is desired to construct general NLs and identify among them those NLs that can be used to introduce forces; it must be pointed out that the NLs for second-order ODEs are known (eg. Olver, 1993; Olver and Sivaloganathan, 1989; Crampin & Saunders, 2005; Vitolo, 1999) and that the NLs constructed using this method are consistent with those previously obtained.

The previous work on defining forces done by Musielak & Vestal et al. (2020), and Vestal & Musielak (2023), is now extended to specific forces that frequently appear in dynamics as well as to nonlinearities that are present in some well-known dynamical systems. For the considered forces and nonlinearities, corresponding gauge functions are given. These gauge functions can be used to obtain the corresponding NLs and to convert undriven and linear dynamical systems into the driven and non-linear systems. In the following, the procedure that allows for this conversion, and uses the Legendre transform to generate new energy terms that are then added to the standard Lagrangian of a given dynamical system, is described, and is then applied to some well-known oscillators in classical dynamics.

Let us generalize the gauge function $\Phi_1(x,t) = c_1 xt$ and consider

$$\Phi_{g1}(x,t) = \sum_{m=1} \sum_{n=1} C_{m,n} x^m t^n \ , \tag{4.26}$$

where $C_{m,n}$ are arbitrary real constants, and m and n are positive integers. Since, for dynamical systems, the variables t and x represent time and displacement, respectively, both variables have dimensions. It is therefore required that the constants $C_{m,n}$

have different dimensions to guarantee that each term in the expansion has the same physical units as the gauge function $[kg\ m^2/s]$. Thus, the dimensions of $C_{1,1} = c_1$ are $[kg\ m/s^2]$, which are of the dimensions of force (see example at the end of Section 2.2).

The general gauge function $\Phi_{g1}(x,t)$ gives the following NL

$$L_{ng1}(\dot{x},x,t) = \sum_{m=1}\sum_{n=1} C_{m,n} \left[m\dot{x}t + nx\right] x^{m-1}t^{n-1} \ , \tag{4.27}$$

where each term of this summation is also a NL.

To further generalize the gauge function $\Phi_{g1}(x,t)$, we consider

$$\Phi_{g2}(x,t) = \sum_{m=1} c_m x^m f_m(t) \ , \tag{4.28}$$

where c_m are arbitrary constants of real values and different dimensions, and m and n are positive integers. In addition, $f_m(t)$ are arbitrary functions that are required to be ordinary ($f_m : \ \mathcal{R} \to \mathcal{R}$) and smooth ($\mathcal{C}^\infty$). The resulting general NL is

$$L_{ng2}(\dot{x},x,t) = \sum_{m=1} \left[m f_m(t)\dot{x}(t) + \dot{f}_m(t)x\right] c_m x^{m-1} \ , \tag{4.29}$$

Once again, we further generalize $\Phi_{g2}(x,t)$ to

$$\Phi_{g3}(x,t) = \sum_{m=1}\sum_{n=1} c_{m,n} f_m(t) g_n(x) \ , \tag{4.30}$$

where $g_n(x)$ are arbitrary functions that are required to be ordinary ($g_n : \ \mathcal{R} \to \mathcal{R}$) and smooth ($\mathcal{C}^\infty$). The gauge function gives the following general NL

$$L_{ng3}(\dot{x},x,t) = \sum_{m=1}\sum_{n=1} c_{m,n} \left[\dot{f}_m(t)g_n(x) + \dot{x}(t)f_m(t)g_n'(x)\right] \ . \tag{4.31}$$

In section 4.2, these general forms for NLs and their corresponding gauge terms are discussed in the context of their implications for systems in physics.

We have now walked through how to construct general gauge functions and their corresponding NLs, which will be used to convert undriven dynamical systems into the driven ones in much of the remainder of the chapter. Later, in section 4.6, this approach will also be used by Vestal & Musielak (2023) to convert linear systems into the nonlinear ones. The main objective of this section was to present the two methods of constructing gauge functions and their corresponding NLs for second-order ordinary differential equations (ODEs) in one dimension. In special cases, the obtained NLs reduce to those found by Musielak et al. (2020) and Musielak & Vestal et al. (2020).

4.2 Gauge Functions and Forces in Dynamics

The main purpose of the NLs constructed in 4.1 is to use them to introduce forces and nonlinearities into otherwise undriven and linear dynamical systems, which has not previously been done. This is achieved by using the Legendre transform to generate new energy terms and adding them to the standard Lagrangian of a dynamical system, which modifies the original system; this is the main novelty of our approach, as described. We also explore physical effects of such modifications and demonstrate that forces and nonlinearities can be introduced to dynamical systems in this way. The presented gauge functions for a variety of known dynamical systems are our main results as these functions can be used to determine the corresponding forces and nonlinearities that are introduced into otherwise undriven and linear dynamical systems.

Let us consider a conservative dynamical system, whose equation of motion is a second-order differential equation that can be obtained from the standard Lagrangian, $L_s(\dot{x}, x)$, that does not depend explicitly on time. However, let us assume that a NL depends explicitly on time. Then, the energy function must be calculated (e.g., Goldstein et al., 2002; José & Saletan, 2002), and, for the total Lagrangian, $L_{tot}(\dot{x}, x, t) = L_s[\dot{x}, x] + L_n(\dot{x}, x, t)$, the energy function is given by

$$E_{tot}(\dot{x}, x, t) = \dot{x}\frac{\partial(L_s + L_n)}{\partial \dot{x}} - (L_s + L_n) = E_s(\dot{x}, x) + E_n(x, t) \ . \tag{4.32}$$

where $E_s(\dot{x}, x)$ represents the total energy of the system, which is equal to its Hamiltonian that can be used to derive the equation of motion from the Hamilton equations (Goldstein et al., 2002; José & Saletan, 2002). The Legendre transformation relates the Lagrangian and Hamiltonian formulations (Abraham & Marsden, 2008), and leads to the same Hamilton equations independently from the null Lagrangian that is added to the standard Lagrangian. The necessary condition is $dE_{tot}/dt = -[\partial(L_s + L_n)/\partial t]$, which plays the same role as the E-L equation does for $L_s(\dot{x})$ (e.g., Goldstein et al., 2002; José & Saletan, 2002; Abraham & Marsden, 2008).

In addition, $E_n(x, t)$ becomes

$$E_n(x, t) = -\frac{\partial \Phi}{\partial t} \ , \tag{4.33}$$

which shows that Φ must depend explicitly on t for $E_n(x, t) \neq 0$. There are known NLs that do not depend explicitly on t but either only on $\dot{x}$ or on both x and $\dot{x}$ including Olver (1993), Olver & Sivaloganathan (1989), and Crampin & Saunders (2005), as all these NLs give $E_n(x) = 0$. In the following we only consider the NLs that depend explicitly on t.

Comparison of Eq. (4.33) to Eq. (4.32) shows that $E_n(x, t)$ and $L_n(\dot{x}, x, t)$ have one common term, and that this term becomes a NL if, and only if, $\Phi \neq \Phi(x)$ and $\Phi = \Phi(t)$, which means that

$$\frac{\partial \Phi}{\partial t} = \frac{d\Phi}{dt} \ . \tag{4.34}$$

However, when $\Phi = \Phi(x, t)$, the total derivative of this gauge function gives a NL, but the resulting $E_n(x, t)$ is not a NL. This is an interesting case as this extra energy term can now be added to, or substracted from, the standard Lagrangian $L_s(\dot{x}, x)$. Then, the following total Lagrangian is obtained

$$L_{tot}(\dot{x}, x, t) = L_s(\dot{x}, x) \pm E_n(x, t) \ . \tag{4.35}$$

Substitution of $L_{tot}(\dot{x}, x, t)$ into the E-L equation gives

$$\frac{d}{dt}\left(\frac{\partial L_s}{\partial \dot{x}}\right) - \frac{\partial L_s}{\partial x} = \pm \frac{\partial E_n}{\partial x} \ . \tag{4.36}$$

If the standard Lagrangian is specified, the equation of motion is obtained from the terms that depend on L_s, and this equation represents an undriven (conservative) dynamical system. However, the presence of the space derivative of E_n is reponsible for converting the conservative equation of motion into the driven one and it can be considered to be a force $F(t)$.

The above results show that the addition of E_n to $L_s(\dot{x}, x)$ modifies the original mechanical system and makes its equation of motion different. The differences are of two kinds, either a time-dependent force appears in the new equation of motion making it an inhomogeneous ODE (driven system), or the form of the original ODE is modified by a nonlinear term. In the first case, the general solutions to both the homogeneous and inhomogeneous ODEs are the same, but the inhomogeneous ODE has also a complimentary solution that accounts for the force. However, in the second case, new solutions must be found independently to each equation of motion with a nonlinear term added, and since the ODEs are nonlinear finding their solutions could be challenging (Kahn 1990).

The force $F(t)$ that appears in the equation of motion is given by

$$F(t) = \pm \frac{\partial E_n}{\partial x} = \pm \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right) \ , \tag{4.37}$$

which means that the initially undriven equation of motion becomes now the driven one. The force depends on the form of the gauge function, $\Phi(x, t)$, which must be an explicit function of both the dependent variable and the independent variable. Since the gauge function $\Phi(x,t)$ can be any differentiable scalar function, the resulting force can also be of any form (see sections 4.4 and 4.5 for additional specific applications).

According to Eq. (4.26), the simplest form of the gauge function $\Phi(x,t)$ that gives a non-zero force is $\Phi_1(x,t) = c_1 x t$, where c_1 is an arbitrary real constant.

Thus, the resulting force is $F_1 = \pm c_1 = \text{const}$, as shown by Musielak et al. (2020), Musielak & Vestal et al. (2020), and Vestal & Musielak (2021). In the following, we generalize this result to gauge functions that depend on arbitrary functions of x and t but keep the variables separated to obtain general analytical results.

The general force resulting from $\Phi_{g1}(x,t)$ (Eq. 4.27) is

$$F_{g1}(x,t) = \sum_{m=1} \sum_{n=1} mn \; C_{m,n} \; x^{m-1} t^{n-1} \; . \tag{4.38}$$

Similarly, for Φ_{g2}, the general force is

$$F_{g2}(x,t) = \sum_{m=1} m \dot{f}_m(t) c_m x^{m-1} \; . \tag{4.39}$$

The resulting force is a power series in the dependent variable with each term of this power being multiplied by any differentiable function of the independent variable. This general force reduces to $F_{g2} = F_1 = c_1 = \text{const}$ if, and only if, $m = 1$ and $f_1(t) = t$.

For Φ_{g3}, and the general force

$$F_{g3}(x,t) = \sum_{m=1} c_{m,n} \dot{f}_m(t) g'_n(x) \; , \tag{4.40}$$

where $g'_n(x)$ is the derivative of $g_n(x)$ with respect to x. The gauge function $\Phi_{g3}(x,t)$ is the most general one that can be considered when the dependent and independent variables are separated. Therefore, the resulting NL $L_{ng3}(\dot{x},x,t)$ and the force $F_{g3}(x,t)$ are also the most general that can be obtained under these conditions, and they are given as infinite sums of all differentiable functions. It must be pointed out that each term in the power series given by Eqs (4.38) and (4.39) and in the summation of functions given by Eq. (4.40) represents a partial NL.

The functions $f_m(t)$ and $g_n(x)$ can be any known elementary functions, including algebraic, trigonometric, exponential, logarithmic, hyperbolic, inverse trigonometric and hyperbolic, and others. The summation of the functions $f_m(t)$ and $g_n(x)$ gives a significant amount of flexibility in defining different forces by using the elementary functions because each term in the summation is a NL that leads to a non-zero force. Since $f_m(t)$ and $g_n(x)$ are to be specified, most known forces in classical mechanics can be formally introduced this way. The coefficients $c_{m,n}$ can be determined by the force required for a given physical system; for instance, the coefficients may represent the amplitude of the force for the given system.

An interesting result is that the above method to define forces can also be extended to introduce nonlinearities into otherwise linear dynamical systems, as will be shown in section 4.6. This shows universality and a broad range of applications of the presented results to classical dynamics.

4.3 From Undriven to Driven Dynamical Systems

Starting from the equations obtained in section 4.1, let us consider specifically how these results apply to a harmonic oscillator system. This application shows how NLs and their respective gauges can be applied to physical systems. The results presented in this section were originally obtained by Musielak & Vestal et al. (2020). For the harmonic oscillator, x(t) is the displacement variable, t is time, and the constants must have different physical dimensions to ensure that the dimensions are physically consistent. The equation of motion of the oscillator is $\hat{D}x(t) = 0$ with $c = k/m$, where k is a spring constant and m is mass. The characteristic frequency of the oscillator is then $\omega_o = \sqrt{c} = \sqrt{k/m}$, and the equation of motion can be written as

$$\ddot{x}(t) + \omega_o^2 x(t) = 0 \ . \tag{4.41}$$

It must be noted that Eq. (4.41) also describes a linear and undamped pendulum if $x(t)$ is replaced by $\theta(t)$, where $\theta(t)$ is an angle of the pendulum, and ω_o is replaced by the pendulum characteristic frequency $\omega_p = \sqrt{c} = \sqrt{g/L}$, where g is gravitational acceleration and L is length of the pendulum. With these replacements, the results by Musielak & Vestal et al. (2020) presented below for the oscillator are also valid for the pendulum.

The standard Lagrangian for this oscillator is

$$L_s[\dot{x}(t), x(t)] = \frac{1}{2}\left[(\dot{x}(t))^2 - \omega_o^2 x^2(t)\right] \ , \tag{4.42}$$

and the total Lagrangian $L_p[\dot{x}(t), x(t)]$ (see Eq. (4.25)) for these harmonic oscillators can be written as

$$L_p[\dot{x}(t), x(t)] = L_s[\dot{x}(t), x(t)] + \frac{d\phi_p}{dt} \ , \tag{4.43}$$

where and the gauge function Φ_p is

$$\Phi_p(t) = \sum_{i=1}^{4} \phi_{pi}(t) \ . \tag{4.44}$$

Recall the general partial gauge functions found in equations 4.17 through 4.20, which lead to the general partial NLs in equations 4.21 through 4.24. For this specific system, these partial gauge functions are

$$\phi_{p1} = \frac{1}{2} C_1 x^2(t) \ , \tag{4.45}$$

$$\phi_{p2} = C_2 x(t) t \ , \tag{4.46}$$

$$\phi_{p3} = C_4 x(t) \ , \tag{4.47}$$

and

$$\phi_{p4} = C_6 t \ . \qquad (4.48)$$

Note that the total derivative of each one of these partial gauge functions gives no contribution to the resulting equation of motion.

Since the gauge functions ϕ_{p2} and ϕ_{p4} depend explicitly on time t, the resulting primary null Lagrangian is also a function of time. Following the formalism presented in section 4.2, we then calculate the primary energy function (e.g., Goldstein et al., 2002; José & Saletan, 2002), E_p, using

$$E_p[\dot{x}(t), x(t)] = \dot{x}\frac{\partial L_p}{\partial \dot{x}} - L_p[\dot{x}(t), x(t)] \ , \qquad (4.49)$$

which gives

$$E_p[\dot{x}(t), x(t)] = \frac{1}{2}\left[(\dot{x}(t))^2 + \omega_o^2 x^2(t)\right] - [C_2 x + C_6] \ , \qquad (4.50)$$

with the first two terms on the RHS representing the energy function E_s for the primary standard Lagrangian and the other two terms corresponding to the primary energy function E_{pf} for the primary gauge function, so that $E_p = E_s + E_{pf}$. Note that using Eq. (4.49) is equivalent to using Eq. (4.32) as this is an application of the formalism developed in the prior sections of this chapter.

In general, $E_p \neq E_{tot}$, with $E_{tot} = E_s = H_s$, where E_{tot} is the total energy of system and H_s is its Hamiltonian, corresponding to the primary standard Lagrangian, and given by $H_s = E_p - E_{pf}$ or

$$H_{ps}[\dot{x}(t), x(t)] = \frac{1}{2}\left[\dot{x}^2(t) + \omega_o^2 x^2(t)\right] \ . \qquad (4.51)$$

Using the Hamilton equations, the equation of motion for the harmonic oscillator given by Eq. (4.41) is obtained. A similar result is derived when the total derivative of

E_p is equal to the negative partial time derivative of L_p that can be written (Goldstein et al., 2002) as

$$\frac{dE_p}{dt} = -\frac{\partial L_p}{\partial t} \ , \tag{4.52}$$

which again gives Eq. (4.41). It must be noted that E_p is a conserved quantity and that $E_p \neq E_{tot}$. This shows that the equation of motion of the harmonic oscillator is also obtained when the energy function is used instead of the primary Lagrangian L_p or the Hamiltonian H_{ps}.

The above results show that among the four primary gauge functions, ϕ_{p1}, ϕ_{p2}, ϕ_{p3} and ϕ_{p4}, the first and third do not contribute to the primary energy function, but the second and fourth do contribute, although each one differently. The partial gauge function ϕ_{p2} breaks into two parts and only the part that depends on $C_2 x$ contributes to the energy function. However, the partial gauge function ϕ_{p4} fully contributes to the energy function. Let us call ϕ_{p2} the *primary F-gauge function*, and ϕ_{p4} the *primary E-gauge function*.

The reasons for these names are as follows. First, the term $C_2 x$ represents energy if, and only if, the coefficient C_2 is a constant acceleration, or a constant force per mass, so that $C_2 x$ is work done by this force on the system. This clearly shows that the primary partial gauge function ϕ_{p2} can be used to introduce forces that cause the constant acceleration. Second, the primary partial gauge function ϕ_{p4} introduces a constant energy shift in the system.

Let us define $F_c = C_2$, where F_c represents a constant acceleration or constant force per mass. Similarly, $E_c = C_6$ is a constant energy shift that could be caused by the force. Then, the primary energy function, which includes both the energy function contribution from the standard Lagrangian and the contributions from the two partial gauge functions, can be written as

$$E_p[\dot{x}(t), x(t)] = \frac{1}{2}\left[(\dot{x}(t))^2 + \omega_o^2 x^2(t)\right] - [F_c x + E_c] \ . \tag{4.53}$$

This demonstrates that some gauge functions can be used to introduce external forces that drive the system but other gauge functions may either generate a shift of the total energy of the system or simply have no effect on the system. In other words, only *gauge functions that depend explicitly on time* may be used to introduce forces in classical mechanics. These are new phenomena caused exclusively by including the gauge functions in classical mechanics.

4.4 Gauge Functions for Bateman Oscillators

In building out the framework of applications of null Lagrangians to physics, the next logical step was to consider if this approach would hold for non-conservative systems. In this section, I show how the formalism can be extended to dissipative systems, through use of the Caldirola-Kanai model for Bateman oscillators. Since these oscillators are non-conservative systems, the derived Lagrangians are not consistent with the Helmholtz conditions (von Helmholtz, 1887) and (Vujanovic & Jones, 1989), which guarantee the existence of Lagrangians for the conservative systems. The Lagrange formalism is developed for this dissipative system of oscillators, which includes both damped and amplified systems. A novel method to derive the Caldirola-Kanai, as introduced in section 3.3, is shown, and its validity in describing the Bateman oscillators is also discussed. In the previous section, the null Lagrangian formalism was developed for conservative systems only, which was done by Musielak & Vestal et al. (2020).

Extending this work to dissipative systems was an important step in building a more complete framework of null Lagrangians and gauge functions in physics, which allows for a broader investigation of physical systems through the lens of null Lagrangians. Further, it allows for a more complete picture of how null Lagrangians and gauges fit into the study of the physical universe. This work was done by Vestal & Musielak (2021).

The CK Lagrangian is modified by taking into account the GFs and it is shown that this modification allows for the conversion of the undriven Bateman oscillators into driven ones, which is our main result. One obtained SL is the CK Lagrangian, which is well-known, and all other NLs and GFs that are derived simultaneously with the CK Lagrangian are new results from Vestal & Musielak (2021).

The Bateman Lagrangian is as given in Eq (3.12) in section 3.3. Let us define $b = \pm\gamma/m$ and $c = k/m = \omega_o^2$, where ω_o is the characteristic frequency of the oscillators, and write Eqs (3.10) and (3.11) as one equation of motion

$$\ddot{x}(t) + b\dot{x}(t) + \omega_o^2 x(t) = 0 \ , \tag{4.54}$$

with the understanding that the damped and amplified oscillators require $b > 0$ and $b < 0$, respectively, and that the variable $x(t)$ describes either damped or amplified oscillator. Let $\hat{D} = d^2/dt^2 + bd/dt + \omega_o^2$ be a linear operator, then Eq. (4.54) can be written in the compact form $\hat{D}x(t) = 0$.

Starting with the following Lagrangian, let us follow the formalism used by Musielak et al. (2020) and Musielak & Vestal et al. (2020) from section 4.1,

$$L_n[\dot{x}_1(t), x_1(t), t] = C_1\dot{x}_1(t)x_1(t) + C_2\left[\dot{x}_1(t)t + x_1(t)\right] + C_4\dot{x}_1(t) + C_6 \ , \tag{4.55}$$

where C_1, C_2, C_4 and C_6 are constants. It is easy to verify that $L_n[\dot{x}_1(t), x_1(t), t]$ is the null Lagrangian, with the constants being arbitrary, and that this NL is constructed to the lowest order of its dynamical variables as shown by Musielak & Vestal et al. (2020). The NL can be added to $L_s[\dot{x}_1(t), x_1(t)]$ without changing the form of the equation of motion resulting from it.

Since the original equations of motion for the Bateman oscillators depend on the dynamical variable $x(t)$ not $x_1(t)$ (see Eqs (3.10) and 3.11), we now use the inverse transform given by Eq. (3.14) to convert the variable $x_1(t)$ into $x(t)$ in both $L_s[\dot{x}_1(t), x_1(t)]$ and $L_n[\dot{x}_1(t), x_1(t), t]$. The resulting total Lagrangian is

$$L[\dot{x}(t), x(t), t] = L_{CK}[\dot{x}(t), x(t), t] + L_n[\dot{x}(t), x(t), t] \ , \tag{4.56}$$

where

$$L_{CK}[\dot{x}(t), x(t), t] = \frac{1}{2}\left[(\dot{x}(t))^2 - \omega_o^2 x^2(t)\right] e^{bt} \ , \tag{4.57}$$

is the Caldirola-Kanai (CK) Lagrangian (see Caldirola, 1941 and Kanai, 1948), derived here independently; comparison of the CK and Bateman Lagrangians (Eqs (4.57) and (3.12)) shows significant differences between them. The presented method gives the following null Lagrangian

$$L_n[\dot{x}(t), x(t), t] = \sum_{i=1}^{3} L_{ni}[\dot{x}(t), x(t), t] \ , \tag{4.58}$$

where the partial null Lagrangians are

$$L_{n1}[\dot{x}(t), x(t), t] = \left(C_1 + \frac{1}{2}b\right)\left[\dot{x}(t) + \frac{1}{2}bx(t)\right] x(t)e^{bt} \ , \tag{4.59}$$

$$L_{n2}[\dot{x}(t), x(t), t] = C_2\left[\left(\dot{x}(t) + \frac{1}{2}bx(t)\right) t + x(t)\right] e^{bt/2} \ , \tag{4.60}$$

and

$$L_{n3}[\dot{x}(t), x(t), t] = C_4\left[\dot{x}(t) + \frac{1}{2}bx(t)\right] e^{bt/2} + C_6 \ , \tag{4.61}$$

where C_1, C_2, C_4, and C_6 are arbitrary but their physical units are different such that all partial null Lagrangians have the same units of energy. These are new null

Lagrangians for the Bateman oscillators. The fact that $L_n[\dot{x}(t), x(t), t]$ and its partial Lagrangians are NLs can be shown by verifying that $\hat{EL}(L_n) = 0$ as well as $\hat{EL}(L_{ni}) = 0$. It must be also noted that $b = 0$ reduces $L_n[\dot{x}(t), x(t), t]$ to the null Lagrangian previously obtained by Musielak & Watson (2020).

After a transformation, which is not relevant to this discussion but can be found in the full paper (ie. Vestal & Musielak, 2021) the standard Lagrangian given in 4.57 becomes

$$L_s[\dot{x}_1(t), x_1(t)] = \frac{1}{2}\left[(\dot{x}_1(t))^2 - \left(\omega_o^2 - \frac{1}{4}b^2\right) x_1^2(t)\right] . \tag{4.62}$$

With $x_1(t)$ as the transformed dependent variable, it gives

$$\ddot{x}_1(t) + \left(\omega_o^2 - \frac{1}{4}b^2\right) x_1(t) = 0 , \tag{4.63}$$

which is the equation of motion for a undamped harmonic oscillator if $b = 0$; then $x_1(t) = x(t)$ (Goldstein et al., 2002) and (José & Saletan, 2002).

The corresponding transformed standard Lagrangian is then

$$L_{CK}[\dot{x}(t), x(t), t] = \frac{1}{2}\left[(\dot{x}(t))^2 - \omega_o^2 x^2(t)\right] e^{bt} , \tag{4.64}$$

which is the Caldirola-Kanai (CK) Lagrangian (Caldirola, 1941 and Kanai, 1948), derived here independently.

For each partial null Lagrangian, the corresponding partial gauge function can be obtained and the results are

$$\phi_{n1}[x(t), t] = \frac{1}{2}\left(C_1 + \frac{1}{2}b\right) x^2(t) e^{bt} , \tag{4.65}$$

$$\phi_{n2}[x(t), t] = C_2 x(t) t e^{bt/2} , \tag{4.66}$$

and

$$\phi_{n3}[x(t), t] = C_4 x(t) e^{bt/2} + C_6 t . \tag{4.67}$$

These partial gauge functions can be added together to form the gauge function

$$\phi_n[x(t), t] = \sum_{i=1}^{3} \phi_{ni}[x(t), t] \ . \tag{4.68}$$

The derived gauge function and partial gauge functions reduce to those previously obtained (Musielak et al., 2020) when $b = 0$ is assumed.

The above partial gauge functions are obtained to the lowest order of the dynamical variables for the Bateman oscillators. Therefore, the only way to generalize the gauge functions given by Eqs (4.65), (4.66) and (4.67), without changing the order of their dynamical variables, is to replace the constants C_1, C_2, C_4 and C_6 with the corresponding functions $f_1(t)$, $f_2(t)$, $f_4(t)$ and $f_6(t)$, which must be continuous and at least twice differentiable, and must depend only on the independent variable t. In order to obey the assumption that our method of constructing null Lagrangians is limited to the lowest order of the dynamical variables, the functions cannot depend on the space coordinates. An interesting result is that the functions give additional degrees of freedom in constructing the null Lagrangians and their gauge functions.

We follow Musielak et al. (2020) and Musielak & Vestal et al. (2020) and call these GFs the general GFs and write them here as

$$\phi_{gn1}[x(t), t] = \frac{1}{2}\left[f_1(t) + \frac{1}{2}b\right] x^2(t) e^{bt} \ , \tag{4.69}$$

$$\phi_{gn2}[x(t), t] = f_2(t) x(t) t e^{bt/2} \ , \tag{4.70}$$

and

$$\phi_{gn3}[x(t), t] = f_4(t) x(t) e^{bt/2} + f_6(t) t \ . \tag{4.71}$$

The general gauge function is obtained by adding these partial gauge functions

$$\phi_{gn}[x(t), t] = \sum_{i=1}^{3} \phi_{gni}[x(t), t] \ . \tag{4.72}$$

The results given by Eqs (4.69) through (4.72) are new general gauge functions for the Bateman oscillators and they generalize those previously obtained for Newton's law of inertia (see Musielak et al., 2020) and for linear undamped oscillators by Musielak & Vestal et al. (2020) for which $b = 0$. The corresponding general null Lagrangians for the Bateman oscillators can easily be obtained by calculating the total derivatives of the derived gauge functions (see Chapter 3).

4.5 Additional Forces in Dynamical Systems

Our theoretical results presented in the previous section demonstrate how NLs and their gauge functions can be used to introduce forces to classical dynamical systems. We consider the equation of motion for a driven harmonic oscillator given by

$$\ddot{x}(t) + x(t) = F(t) \ , \tag{4.73}$$

where typical forms of the force $F(t)$ are given in Table 4.1. The presented forces are time-dependent and they are given by some well-known elementary functions.

The aim is now to identify gauge functions that can be used to introduce these forces. For the forces presented in Table 4.1, the most appropriate is the general gauge function given by Eq. (4.30). The procedure is straightforward and requires comparing the forces in Table 4.1 to the general force term given by Eq. (4.40). In other words, we identify the coefficients $c_{m,n}$ and the derivatives of functions $f_m(t)$ and $g_n(x)$. Then, Eq. (4.30) can be used to determine the gauge function corresponding to each force of Table 4.1 as well as the NL corresponding to each force, which can be obtained using Eq. (4.31).

We may also consider F that does not explicitly depend on t but instead depends on x. Taking $F(x) = -\varepsilon F_0 x$, where $0 < \varepsilon \leq 1$, and substituting into Eq. (4.40), we

Dynamical systems	Force $F(t)$	Gauge function $\Phi(x,t) = \Phi_1(x,t) + \Phi_2(x,t)$
Driven oscillators	$F(t) = F_0 \cos t$	$\Phi_1(x,t) = xF_0 \sin t$ $\Phi_2(x,t) = 0$
	$F(t) = F_0 \cos^2 t$ $= \frac{1}{2} F_0 (1 + \cos 2t)$	$\Phi_1(x,t) = \frac{1}{2} xtF_0$ $\Phi_2(x,t) = \frac{1}{4} xF_0 \sin 2t$
	$F(t) = F_0 \cos^3 t$ $= \frac{1}{4} F_0 (3 \cos t + \cos 3t)$	$\Phi_1(x,t) = \frac{3}{4} xF_0 \sin t$ $\Phi_2(x,t) = \frac{1}{12} xF_0 \sin 3t$
	$F(t) = F_0 e^{it}$	$\Phi_1(x,t) = -ixF_0 e^{it}$ $\Phi_2(x,t) = 0$
	$F(t) = F_1 \cos t$ $+F_2 \sin t$	$\Phi_1(x,t) = xF_1 \sin t$ $\Phi_2(x,t) = -xF_2 \cos t$
RLC circuits	$E(t) = E_0 \sin t$	$\Phi_1(x,t) = -xE_0 cost$ $\Phi_2(x,t) = 0$

Table 4.1. Selected forces in dynamical systems and their gauge functions, from Goldstein et al. (2002), José et al. (2002), and Kahn (1990)

obtain the equation of motion of the altered simple harmonic oscillator (Musielak et al., 2020). The gauge function for this force is $\Phi = -\frac{\varepsilon}{2} F_0 x^2 t$ as obtained from Eq. (4.30), and the corresponding NL is $L_n(\dot{x}, x, t) = -\frac{1}{2} \varepsilon F_0 (2\dot{x}t + x)x$.

The gauge functions presented in Table 4.1 become special cases of general gauge functions given by Eqs (4.26), (4.28), and (4.30). As an example, let us consider the force $F(t) = \cos^2 t = \frac{1}{2} F_0 (1 + \cos 2t)$. Then, its gauge function $\Phi_1(x,t) = \frac{1}{2} xtF_0 = \Phi_{g1}(x,t)$ with $m = n = 1$ and $C_{1,1} = \frac{1}{2} F_0$ (see Eq. 4.64). However, the gauge function $\Phi_2(x,t) = \frac{1}{4} xF_0 \sin 2t = \Phi_{g2}(x,t)$ with $m = 1$ and $C_1 = \frac{1}{4} F_0$ and $f_1(t) = \sin 2t$ (see Eq. 5.22). The same force can also be identified as $\Phi_{g1}(x,t) = C_{1,1} f_1(t) g_1(x) +$

$C_{2,2} f_2(t) g_2(x)$, where $C_{1,1} = \frac{1}{2} F_0$, $f_1(t) = t$, $g_1(x) = x$, $C_{2,2} = \frac{1}{4} F_0$, $f_2(t) = \sin 2t$ and $g_2(x) = x$. Similar identification can be performed for all forces given in Table 4.1.

In the above results, the gauge functions were obtained for forces that are typically used to drive harmonic oscillators. However, the process can be reversed and forces can be determined by using the gauge functions presented in sections 4.1.1 and 4.1.2. The same method can also be used to introduce nonlinearities to the equation of motion of a harmonic oscillator, which will be shown in the section immediately following.

4.6 Gauge Functions and Nonlinear Dynamics

Using the method developed in section 4.2, herein I show how nonlinearities can be introduced to a system, in a similar way to how this was done for forces; the results are presented for these nonlinearities in Table 4.2, paralleling the presentation of forces in Table 4.1. It must be noted that we make no claim that our method can only be used for forces and nonlinearities, and further exploration of how powerful NLs and GFs are for systems in science and engineering is a promising area of future research.

Having shown how forces can be added to the equations of motion for harmonic oscillators and the Bateman oscillators in the previous chapter, it will now be shown how the method can be applied to introduce nonlinearities. In this way, a well-known nonlinear equation is produced. This is important for i) showing a new way to introduce nonlinear terms, and ii) demonstrating the potential of the NL formalism for introducing a range of physical phenomena; this was shown by Vestal & Musielak (2023).

First, we started with the following equation of motion,

$$\ddot{x}(t) + x(t) = H(\dot{x}, x) \ , \tag{4.74}$$

from which it is possible to represent different forms of nonlinearities in dynamical systems by a choice of $H(\dot{x}, x)$. We selected several examples of nonlinear dynamical systems, including well-known systems such as the Duffing oscillator (see Levy-Leblond, 1969 and Musielak et al., 2020), and from H(x) identified the gauge function ($\Phi(x,t)$) that produces each one; this was done by following the approach used to similarly match forces to their generating GFs, as presented in section 4.5. The results for these GFs for nonlinearities are presented in Table 4.2.

The gauge functions presented in Table 4.1 become special cases of general gauge functions given by Eqs (4.26), (4.28) and (4.30). It is straightforward to show that any $\Phi(x,t)$ in Table 4.2 can be identified as either $\Phi_{g1}(x,t)$, $\Phi_{g2}(x,t)$, or $\Phi_{g3}(x,t)$ from section 4.1 by an making an appropriate selection of the coefficients and functions in these equations. Further, note that more general nonlinearities can be obtained from this method than the results presented in Table 4.2.

All dynamical systems considered in these two tables illustrate the relationships between forces and nonlinearities and their corresponding gauge functions. The purpose of this illustration is to demonstrate that typical forms of forces and nonlinearities have gauge functions that can be used to introduce them by way of the method presented in the earlier section. Further, this method of introducing nonlinearities into otherwise linear equations of motion can be combined with the method of defining forces so that linear, undriven dynamical systems can be converted into nonlinear and driven ones; this means that our method can be applied to a broad range of dynamical systems.

Type of oscillator	Nonlinearity $H(x)$	Gauge function $\Phi(x,t)$
Quadratic	$H(x) = -\varepsilon x^2$	$\Phi(x,t) = -\frac{1}{3}\varepsilon x^3 t$
Duffing	$H(x) = -\varepsilon x^3$	$\Phi(x,t) = -\frac{1}{4}\varepsilon x^4 t$
Quadratic and cubic	$H(x) = -\varepsilon(x^2 + x^3)$	$\Phi(x,t) = -\frac{\varepsilon}{3}x^3 t - \frac{\varepsilon}{4}x^4 t$
Quartic	$H(x) = -\varepsilon x^4$	$\Phi(x,t) = -\frac{1}{5}\varepsilon x^5 t$
Quintic	$H(x) = -\varepsilon x^5$	$\Phi(x,t) = -\frac{1}{6}\varepsilon x^6 t$
Higher-order	$H(x) = -\varepsilon x^{2n+1}$	$\Phi(x,t) = -\frac{\varepsilon}{2n+2}x^{2n+2} t$

Table 4.2. Selected nonlinearities in dynamical systems, from Goldstein et al. (2002), José et al. (2002), and Abraham et al. (2008), and their gauge functions.

Thus, the presented results demonstrate a new important role of gauge functions and null Lagrangians in classical mechanics and, specifically, in its theory of dynamical systems. It is natural at this point to ask what other physical phenomena, beyond nonlinearities and forces, might similarly have a gauge function representation.

4.7 Gauge Functions for Equations with Special Function Solutions

It is possible to write a Lagrangian general enough that it can produce multiple SFs by a simple choice of variables. In this section, I will discuss how we developed a method to derive general standard Lagrangians and NLs for SFs, starting with the aforementioned Lagrangian, which is given below. This approach may feel familiar, as the approach of starting with a general equation to describe a range of systems was similarly utilized in the previous section. For this work, an approach like the ones described in section 4.1 was followed to constuct the NLs and GFs. The SLs investigated here depend on the square of the first derivative of the dependent variable (kinetic energy-like term) and the square of the dependent variable (potential energy-like term). This project was also the impetus for the prior work that resulted in a generalized formalism for introducing forces and nonlinear terms from NLs, which was described in prior sections. Once these NLs and GFs were shown to reproduce the equations for SFs, I felt it was important to consider the full range of NLs that could be added to a SL.

Special functions play an important role in the mathematical framework of physics, as discussed in section 3.4, and exploring the deeper link between these equations and the corresponding gauge functions from which they can be derived was the motivation for the work described in this section. As some SFs are of particular interest for applications to physical systems, a focus was given to these systems; however, the work described herein could be expanded in a straightforward way to consider other SFs of interest. Our results are applied to the Bessel, Hermite, and Legendre equations, as these specific SFs are used in many physical applications. The choice to focus on these specific ODEs was made due to their many physical applications familiar to graduate and undergraduate science students, and the presented results should be of interest to physicists, applied mathematicians, and engineers.

Prior to this project, the role of NLs and GFs for ODEs with special function solutions had not yet been explored. This section will describe the work that I have done in exploring this link (see Dange, Vestal, & Musielak, 2021).

The process of determining the link between a given Lagrangian and the generating GF was done first for the general Lagrangian, and the resulting general gauge functions and null Lagrangians were used to solve for the coefficients B(x) and C(x) for the three specific SFs. This process is shown below.

Let $\widetilde{D} = d^2/dx^2 + B(x)d/dx + C(x)$ be a linear operator operating on smooth C^∞ functions. If $\widetilde{D}$ acts on y(x), which is also ordinary and smooth, then the resulting ODE can be written in the following explicit form

$$y"(x) + B(x)y'(x) + C(x)y(x) = 0 \tag{4.75}$$

As aforementioned, by specifying the coefficients B(x) and C(x), which is done here, all ODEs with the special function solutions are obtained; for these equations, we derive the SLs and NLs in this way.

Then, let us consider the general Lagrangian

$$L(y', y(x), x) = \frac{1}{2}f_1(x)y'^2 + \frac{1}{2}f_2(x)y(x)y'(x) + \frac{1}{2}f_3(x)y^2, \tag{4.76}$$

where $f_1(x)$, $f_2(x)$, and $f_3(x)$ are ordinary and smooth functions to be determined. Note specifically that this Lagrangian chosen depends on the square of the first derivative of the dependent variable (kinetic energy-like term), the square of the dependent variable (potential energy-like term) and on the mixed term with the dependent variable and its derivative. Substituting the above Lagrangian into the E-L equation given in Eq. 2.1 yields

$$y"(x) + \left(\frac{f_1'}{f_2'}\right) y'(x) + \frac{1}{f_1}\left(\frac{1}{2}f_2' - f_3\right) y(x) = 0. \tag{4.77}$$

Next, by comparing equations 4.75 and 4.77, B(x) and C(x) can be determined and are found to be $B(x) = \frac{f_1'}{f_1}$ and $C(x) = \frac{1}{f_1}\left(\frac{1}{2}f_2' - f_3\right)$.

By considering Eq. (4.76) and solving for functions f_1 and f_3, we then find that

$$f_1 = c_1 e^{\int B(x)dx} = c_1 E_s \tag{4.78}$$

$$f_3 = \frac{1}{2}f_2' - C(x) \cdot f_1 = \frac{1}{2}f_2' - C(x) \cdot (c_1 E_s), \tag{4.79}$$

where c_1 is the integration constant and $E_s = e^{\int B(x)dx}$.

Substituting f_1 and f_3 into the $L(y', y, x)$, we find

$$L_{eq}(y', y, x) = L_s(y', y, x) + L_n(y', y, x) \tag{4.80}$$

where

$$L_s(y', y, x) = \frac{1}{2} c_1 E_s \left[y'(x)^2 - C(x) y(x)^2 \right] \tag{4.81}$$

and

$$L_n(y', y, x) = \frac{1}{2} y [f_2 y'(x) + \frac{1}{2} f_2'(x) y(x)], \tag{4.82}$$

with $L(y', y, x)$ being a combination of the general standard Lagrangian $L_s(y', y, x)$ and the general null Lagrangian, $L_n(y', y, x)$. It must be noted that $L_s(y', y, x)$ generalizes the Caldirola-Kanai (CK, see sections 3.3, 4.4) Lagrangian and it reduces to the CK Lagrangian when $B(s) = b = const$ and $C(x) = c = const$; the standard Lagrangian also describes a harmonic oscillator with time dependent mass and a spring constant.

Having obtained the general null Lagrangian, we now derive the corresponding general form of the gauge function (Φ) by using,

$$L_{null} = \frac{1}{4} f_2'(x) y^2(x) + \frac{1}{2} f_2(x) y(x) y'(x) = \frac{d\Phi}{dx}. \tag{4.83}$$

The gauge function for Eq. (4.83) is obtained,

$$\Phi = \frac{1}{4} f_2(x) y^2(x). \tag{4.84}$$

Further, as $f_2(x)$ is arbitrary, three cases were considered. The trivial case of $f_2 = 0$ leads to the minimum L_s. The contribution from the null Lagrangian vanishes for this case, and $\Phi = 0$. The gauge obtained by substituting this case into the GF is $\Phi = \frac{1}{4} f_2(x) y^2(x) = 0$ (no gauge function).

For the case of a constant value of f_2, $L_{eq,mid} = L_{s,max} + L_{n,max}$. The resulting gauge equation, again obtained via Eq. (4.84) is $\phi = \frac{1}{4} f_2(x) y^2(x) = \frac{1}{4} c y^2(x) = c_2 y^2(x)$ (variable gauge function). The third case is $f_2 = f_1'$, which leads to the maximum value for the equivalent Lagrangian, $L_{eq,max} = L_{s,max} + L_{n,max}$. The corresponding maximum variable gauge function is then obtained similarly to the two aforementioned and is $\phi = \Phi_{max} = \frac{1}{4} c_1 E_s B(x) y^2(x)$. Applications of these results to the three selected ODEs, along with the general result, are given in Table 4.3. Note that the choice of α, β, and γ allow for different Bessel equations (regular, modified, spherical, and spherical modified) to be obtained.

To summarize, in this project general standard and null Lagrangians were found for linear second-order ODEs whose solutions were given by the SFs of mathematical physics and appear published in Dange, Vestal & Musielak (2021). The derived gauge functions are also a new result. The obtained results were applied to the Bessel, Hermite, and Legendre equations, making them very relevant to physicists and applied mathematicians; further, these results could easily be applied to any ODE with SF solutions, presenting a promising area of future work.

4.8 Summary

This chapter presents a fundamental first step in constructing NLs and exploring their new important roles in classical physics. The chapter begins with two methods to construct standard null Lagrangians. This is followed by my first published results that appeared in the paper by Musielak & Vestal et al. (2020), wherein we present an approach to convert undriven oscillators to driven ones in a novel way by using the derived NLs. The formalism was then extended to dissipative systems, focusing on the Bateman oscillator system, by Vestal & Musielak (2021). This work showed that application of the formalism developed is not limited to conservative systems,

TABLE 1.

	Equation	$L = L_S + L_n$	$L_{n,max}$	φ_{max}
General	$y'' + B(x)y' + C(x)y = 0$	$\frac{1}{2}f_1(x)y'^2 + \frac{1}{2}f_2(x)yy' + \frac{1}{2}f_3(x)y^2$	$\frac{1}{2}y\left[f_2y' + \frac{1}{2}f_2'y\right]$	$\frac{1}{4}f_2(x) \cdot y^2(x)$
General Bessel	$B(x) = \frac{\alpha}{x}$ $C(x) = \beta\left(1 + \gamma\frac{\mu^2}{x^2}\right)$	$\frac{1}{2}c_1 \cdot x\left[y'^2 - C(x)y^2\right] \frac{1}{+2}y\left[f_2y' + \frac{1}{2}f_2'y\right]$	$\frac{1}{2}c_1e^{\alpha c_2} \cdot \alpha x^{\alpha-1} \cdot y\left[\frac{\alpha-1}{2}x^{-1}y + y'\right]$	$\frac{1}{4}c_1 \cdot e^{\alpha c_2} \cdot \alpha x^{\alpha-1} \cdot y^2(x)$
Regular Hermite	$B(x) = -x$ $C(x) = x$	$\frac{1}{2}c_1e^{\frac{-x^2}{2}}\left[y'^2 - xy^2\right] + \frac{1}{2}y\left[f_2y' + \frac{1}{2}f_2'y\right]$	$\frac{1}{2}e^{\frac{-x^2}{2}} \cdot y\left[\frac{1}{2}y(x^2-1) - xy'\right]$	$\frac{-1}{4}e^{\frac{-x^2}{2}} \cdot x \cdot y^2(x)$
Regular Legendre	$B(x) = \frac{2x}{1-x^2}$ $C(x) = \frac{l(l+1)}{1-x^2}$	$\frac{1}{2}c\left\|1-x^2\right\|\left[y'^2 - \left(\frac{l(l+1)}{1-x^2}\right) \cdot y^2\right] + \frac{1}{2}y\left[f_2y' + \frac{1}{2}f_2'y\right]$	$\pm xcyy' \pm \frac{1}{2}cy^2$	$\pm\frac{1}{2}c \cdot x \cdot y^2(x)$

Figure 4.1. Selected equations with special function solutions and their corresponding gauge functions. By the choice of α , β and γ, different (regular, modified, spherical and spherical modified) Bessel equations are obtained (Dange, Vestal, & Musielak 2021).

which is a novel result. I also developed a new approach to finding NLs and their gauge functions in a generalized way, and for going between NLs and corresponding gauge functions. The formalism was developed for introducing forces and nonlinearities directly from these gauges, and I showed how forces and nonlinearities of interest for applications to physics can be reproduced using this method in Vestal & Musielak (2023). The successful adaption of the NL formalism to nonlinearities further demonstrates how promising the study of NLs and gauge functions is for physical systems. There may be additional phenomena beyond those described herein that NLs can also be used to introduce to a system. Moreover, standard and null Lagrangians, and their corresponding gauge functions, were also derived for linear second-order ODEs whose solutions are given by SFs of mathematical physics and appeared published in Dange, Vestal, & Musielak, 2021. The derived gauge functions are also a new result. The research presented herein helps to lay the groundwork for a large-scale investigation of the application of NLs to the field of physics, of which this dissertation is a piece.

CHAPTER 5

Non-standard Null Lagrangians and their Applications to Dynamics

In this chapter, I will present additional applications of non-standard and non-standard null Lagrangians (NSNLs) to systems of interest in dynamics. I start by describing a method to construct NSNLs. This will be followed by first presenting the way in which NSNLs and their corresponding non-standard gauge functions (NSGFs) can be used to introduce dissipative forces to Classical Dynamics, extending what was done in Chapter 4 to non-standard Lagrangians, which is the main conclusion of work by Segovia, Vestal, & Musielak (2022); this work illustrates the unique role that NSNLs play in physics, notably that the behavior of the non-standard representation of a system can differ from the behavior of the standard Lagrangian. Further, in this project we showed that the NSNL for the law of inertia is Galilean invariant but the forces resulting from the NSNL are not Galilean invariant, except for the special case of only time-dependent forces. The chapter concludes with recent work of Das & Musielak (2023), who demonstrated strong relationships between standard and non-standard null Lagrangians and non-standard Lagrangians.

A method to construct NSNLs and corresponding NSGFs is presented in section 5.1. An extension of the formalism for introducing forces by way of NLs and GFs to consider NSNLs and NSGFs, is their Galilean invariance, are given in sections 5.2 and 5.3. Nonlinearities arising from GFs are discussed in section 5.4. Newly discovered relationships between non-standard Lagrangians and standard and non-standard null Lagrangians are discussed in section 5.5.; the chapter is summarized in section 5.6.

5.1 Methods to Construct Non-standard Null Lagrangians

The next step in investigating the role of NLs in physics is to explore if a similar formalism can be found for non-standard Lagrangians (see section 2.3.2) to find a non-standard null Lagrangian (NSNL). No such formalism exists within literature.

Recall that NSLs are described as Lagrangians different from standard Lagrangians (Musielak, 2021). Is it possible to write a non-standard null Lagrangian, what would a term like this look like, and what sort of physical significance might it have, if any? The first two of these questions will be addressed in this section. An approach to finding NSNLs (non-standard Lagrangians that also satisfy the two conditions for being a null Lagrangian) is now presented.

We will begin by considering a very commonly used NSL,

$$L_{ns}(\dot{x}, x, t) = \frac{1}{g_1(t)\dot{x} + g_2(t)x + g_3(t)} \ , \tag{5.1}$$

where $g_1(t)$, $g_2(t)$ and $g_3(t)$ are arbitrary and differentiable functions that are to be determined. In addition to being different from standard Lagrangians, non-standard Lagrangians must be similar to this equation (Musielak, 2021). The NSL, and therefore the NSNL, must then contain $\dot{x}$, x and arbitrary functions of t, or constants. Further, to be a NSNL, it is necessary that the power of the dependent variable and its derivative not exceed their order as given in Eq. (5.1).

Proposition 1: Let a_1, a_2, a_3 and a_4 be constants in the following non-standard test-Lagrangian

$$L_{ns,test1}(\dot{x}, x, t) = \frac{a_1 \dot{x}}{a_2 x + a_3 t + a_4} \ . \tag{5.2}$$

Then, $L_{ns,test1}(\dot{x}, x, t)$ is a null Lagrangian if, and only if, $a_3 = 0$.

Proof: Since this Lagrangain must satisfy the E-L equation, $\hat{EL}\{L_{ns,test1}(\dot{x}, x, t)\} = 0$, the required condition is $a_1 a_3 = 0$. With $a_1 \neq 0$, then $a_3 = 0$, and $L_{ns,test1}(\dot{x}, x, t) = L_{nsn1}(\dot{x}, x, t)$, where the latter is the non-standard NL. This concludes the proof.

Corollary 2: Let $L_{nsn1}[\dot{x}, x]$ be the non-standard null Lagrangian given by

$$L_{nsn1}(\dot{x}, x, t) = \frac{a_1 \dot{x}}{a_2 x + a_4} \ , \tag{5.3}$$

then, its gauge function $\Phi_{nsn1}(x)$ is

$$\Phi_{nsn1}(x) = \frac{a_1}{a_2} \ln |a_2 x + a_4| \ . \tag{5.4}$$

Corollary 3: Another non-standard NL that can be constructed is $L_{nsn2}(t) = b_1/(b_2 t + b_3)$ with its gauge function $\Phi_{nsn2}(t) = (b_1/b_2) \ln |b_2 t + b_3|$; however, this Lagrangian and its gauge function do not obey the first condition, thus, they will not be further considered.

Generalization of $L_{nsn1}(\dot{x}, x)$ is now presented in Proposition 2.

Proposition 2: Let $h_1(t)$, $h_2(t)$ and $h_4(t)$ be at least twice differentiable functions, and $L_{nsn1}(\dot{x}, x)$ be the non-standard NL given by Eq. (5.3) with its non-standard GF given by Eq. (5.4). A more general non-standard NL is obtained if, and only if, the constants in $\Phi_{nsn1}(x)$ are replaced by the corresponding functions $h_1(t)$, $h_2(t)$ and $h_4(t)$.

Proof: Replacing the constant coefficients a_1, a_2 and a_4 in $L_{nsn1}(\dot{x}, x)$ by the functions $h_1(t)$, $h_2(t)$ and $h_4(t)$, respectively, the resulting Lagrangian is

$$L_{ns,test2}(\dot{x}, x, t) = \frac{h_1(t)\dot{x}}{h_2(t)x + h_4(t)} \ . \tag{5.5}$$

Using $\hat{EL}\{L_{ns,test2}(\dot{x}, x, t)\} = 0$, it is seen that $L_{ns,test2}(\dot{x}, x, t)$ is the non-standard NL only when $h_1(t) = a_1$, $h_2(t) = a_2$ and $h_4(t) = a_4$, which reduces $L_{ns,test2}(\dot{x}, x, t)$ to $L_{nsn1}(\dot{x}, x, t)$, and shows that no generalization of $L_{nsn1}(\dot{x}, x, t)$ can be accomplished this way.

Now, replacing the constant coefficients in $\Phi_{nsn1}(x)$ by the functions $h_1(t)$, $h_2(t)$ and $h_4(t)$ generalizes the gauge function to

$$\Phi_{nsgn}(x) = \frac{h_1(t)}{h_2(t)} \ln |h_2(t)x + h_4(t)| \ . \tag{5.6}$$

Since the total derivative of any differentiable scalar function that depends on x and t is a null Lagrangian, the following non-standard NL is obtained

$$L_{nsgn}(\dot{x}, x, t) = \frac{h_1(t)[h_2(t)\dot{x} + \dot{h}_2(t)x] + \dot{h}_4(t)}{h_2(t)[h_2(t)x + h_4(t)} + \left[\frac{\dot{h}_1(t)}{h_2(t)} - \frac{h_1(t)\dot{h}_2(t)}{h_2^2(t)}\right] \ln|h_2(t)x + h_4(t)| \,. \tag{5.7}$$

As expected $\hat{EL}\{L_{nsgn}(\dot{x}, x, t)\} = 0$. Therefore, $L_{nsgn}(\dot{x}, x, t)$ is the general (when compared to Eq. 5.3) non-standard null Lagrangian. This concludes the proof.

In this way, a new family of NSNLs, given in a general form by $L_{nsgn}(\dot{x}, x, t)$, is obtained. Further, the corresponding gauge functions for these NSNLs are found to be $\Phi_{nsgn}(\dot{x}, x, t)$. In section 3.1, a physical application of NSNLs to the law of inertia is presented and a remarkable result arising only from the non-standard form is discussed.

5.2 Non-standard Null Lagrangians and Forces in Classical Mechanics

In this section, the role of NSNLs and their NSGFs in defining forces in CM is investigated by following Segovia, Vestal, & Musielak (2022). The main result found from this work is that dissipative forces can be introduced to CM by using the NSGFs. The relationships between the forces and types of null Lagrangians that introduce them are used to gain novel insight into the physical meaning of the standard and non-standard null Lagrangians and the differences between the two. In section 5.3, the obtained results are further discussed as they relate to Newton's laws, which were introduced in NSL form in section 3.1.2. While standard and non-standard Lagrangians are known for producing the same equation of motion for a given system, we explore a unique aspect of the non-standard formulation that leads to a compelling result.

As discussed in section 2.3.2, non-standard Lagrangians are a lesser known family of Lagrangians in which neither explicit kinetic nor potential energy-like terms are present. These Lagrangians have been introduced to CM in recent years. However, the physical meaning of the NSLs remains unclear (see discussion in section 2.3.2). In this section, I present an extension to the formalism established for SLs to NSLs.

The forms of SLs and NSLs are distinct, thus, there are also the corresponding NLs whose forms resemble those two families of Lagrangians. In the previous work on NLs (e.g., Musielak & Vestal et al., 2020; Vestal & Musielak 2021, 2023; Jammer, 1997, 2000), the so-called standard NLs have been used. Recently, non-standard NLs have been introduced (Musielak, 2021), and herein the aim is to understand the physical meaning of the non-standard NLs and their role in dynamical systems.

The results herein demonstrate that the non-standard NLs can be used to introduce dissipative forces, and that this distinguishes them from the standard NLs that are responsible for non-dissipative forces. From a physical point of view, this means that the standard Lagrangians are more suitable for describing undamped dynamical systems, while the non-standard Lagrangians are more applicable to damped systems; this statement applies equally to the SLs and NSLs as well as to the standard and non-standard NLs. Further, in section 5.3 the obtained results are used to introduce dissipative forces to the law of inertia and convert it into Newton's second law; Galilean invariance of the laws and their Lagrangians is also investigated and discussed.

A method to construct NSLs introduced in section 5.1 can now be used to find

$$L_n(\dot{x}, x) = \frac{a_1 \dot{x}}{a_2 x + a_4}, \tag{5.8}$$

where a_1, a_2 and a_4 are arbitrary constant coefficients. It is easy to verify that $L_n(\dot{x}, x)$ is indeed a null Lagrangian, and that its gauge function is

$$\Phi_n(x) = \frac{a_1}{a_2} \ln |a_2 x + a_4| \ . \tag{5.9}$$

To generalize these results, it was suggested that the coefficients a_1, a_2 and a_4 be replaced by the corresponding functions of t so the generalized gauge function [31] becomes

$$\Phi_{gn}(x, t) = \frac{h_1(t)}{h_2(t)} \ln |h_2(t) x + h_4(t)| \ , \tag{5.10}$$

where $h_1(t)$, $h_2(t)$ and $h_3(t)$ are twice differentiable but otherwise arbitrary functions of t, and in addition $h_2 \neq 0$. This gauge function gives the following general non-standard NL

$$L_{gn}(\dot{x}, x, t) = \frac{h_1(t)[h_2(t)\dot{x} + \dot{h}_2(t)x] + \dot{h}_4(t)}{h_2(t)[h_2(t)x + h_4]} + \left[\frac{\dot{h}_1(t)}{h_2(t)} - \frac{h_1(t)\dot{h}_2(t)}{h_2^2(t)} \right] \ln |h_2(t)x + h_4| \ , \tag{5.11}$$

which is significantly different than $L_n(\dot{x}, x)$ given by Eq. (5.8), despite the fact that both are null Lagrangians.

Let $\Phi_{null}[x(t), t]$ be a gauge function (either $\Phi_n(x)$ or $\Phi_{gn}(x, t)$). Any NL can then be expressed as $L_{null}(\dot{x}, x, t) = d\Phi_{null}(x, t)/dt$, which gives the energy function

$$E_{null}(\dot{x}, x, t) = -\frac{\partial \Phi_{null}(x, t)}{\partial t} \ . \tag{5.12}$$

By using $\Phi_{null}(x, t) = \Phi_n(x)$, we find $E_n = 0$ because the gauge function does not depend explicitly on time. However, for $\Phi_{null}(x, t) = \Phi_{gn}(x, t)$, we obtain

$$E_{gn}(\dot{x}, x, t) = -\left[\frac{\dot{h}_1(t)}{h_1(t)} - \frac{\dot{h}_2(t)}{h_2(t)} \right] \Phi_{gn}(x, t) - \left[\frac{h_1(t)}{h_2(t)} \right] \frac{h_2(t)\dot{x} + \dot{h}_2(t)x + \dot{h}_4(t)}{h_2(t)x + h_4(t)} \ . \tag{5.13}$$

This shows that different gauge functions have different effects on the energy function, namely, for some it can be zero, but for others becomes non-zero, and it is interesting that it depends on the gauge function itself.

Having obtained the relationship between the energy function and the gauge function (Eq. 5.12), and knowing that $E_{null}(\dot{x}, x, t)$ resulting from any non-standard NL is not a NL by itself, we may add this extra term to the NSL given by Eq. (5.1). This gives

$$L_t(\dot{x}, x, t) = L_{ns}(\dot{x}, x, t) - \frac{\partial \Phi_{null}(x, t)}{\partial t} \ , \tag{5.14}$$

where the NSL is either

$$L_{ns}(\dot{x}, x, t) = \frac{1}{C_1 f^2(t)[f(t)\dot{x} - a_o x + C_2]} \ , \tag{5.15}$$

or

$$L_{ns}(\dot{x}, x, t) = \frac{1}{C_3 \dot{x}} \ , \tag{5.16}$$

with $f(t) = a_o t + v_o$, and $\Phi_{null}(x, t) = \Phi_{gn}(x, t)$, since there is no contribution from $\Phi_n(x)$.

The equation of motion resulting from Eq. (5.15) is

$$\frac{2\ddot{x}}{C_1[f(t)\dot{x} - a_o x + C_2]^3} = F(\dot{x}, x, t) \ , \tag{5.17}$$

and using Eq. (5.16), we obtain

$$\frac{2\ddot{x}}{C_3 \dot{x}^2} = F(\dot{x}, x, t) \ , \tag{5.18}$$

where the forcing function is

$$F(\dot{x}, x, t) = -\frac{\partial}{\partial x}\left[\frac{\partial \Phi_{gn}(x, t)}{\partial t}\right] = \frac{\partial E_{gn}(\dot{x}, x, t)}{\partial x} \ . \tag{5.19}$$

Because of the presence of additional terms on the LHS of the above equations, we may write Eqs (5.17) and (5.18) in the following forms: $\ddot{x} = F_1(\dot{x}, x, t)$ and

$\ddot{x} = F_2(\dot{x}, x, t)$, respectively, where $F_1(\dot{x}, x, t) = F(\dot{x}, x, t)C_1[f(t)\dot{x} - a_o x + C_2]^3$ and $F_2(\dot{x}, x, t) = F(\dot{x}, x, t)C_3\dot{x}^2/2$. The effects of these extra terms on the forcing function are discussed in Section 5.

The above results demonstrate how classical forces can be defined using the non-standard NLs, and also show how the law of inertia can be converted into the second law of dynamics. The main difference between the previous results (eg. Musielak & Vestal et al., 2020; Vestal & Musielak, 2021) and the ones presented in this section is the physical nature of the forces introduced by the non-standard NLs, namely, the forces resulting from the non-standard NLs are dissipative, while the forces introduced by the standard NLs are non-dissipative.

Our method of converting the first law of dynamics into the second one gives an independent way to introduce forces in CM, and supplements Newton's definition of forces that directly relates them to object's acceleration and mass (e.g., Jammer, 1997, 2000). Thus, the presented results show a deeper connection between Newton's first and second laws of dynamics, and demonstrate that non-standard NLs can be used to turn undriven dynamical systems into driven ones.

Using Eqs (5.13) and (5.19), we obtain the explicit form of the forcing function

$$F(\dot{x}, x, t) = \frac{h_1(t)h_2(t)}{[h_2(t)x(t) + h_4(t)]^2}\left[\dot{x} + \left(\frac{\dot{h}_2(t)}{h_2(t)} - \frac{\dot{h}_1(t)}{h_1(t)}\right)x\right] - \frac{h_1(t)h_4(t)}{[h_2(t)x(t) + h_4(t)]^2}\left(\frac{\dot{h}_4(t)}{h_4(t)} - \frac{\dot{h}_1(t)}{h_1(t)}\right) . \tag{5.20}$$

For $F(\dot{x}, x, t)$ to be zero, either $h_1(t) = 0$ or $h_1(t) = h_2(t) = 0$. There are several special cases, like $h_1(t) = c_1$, $h_2(t) = c_2$, and $h_4(t) = 0$, which gives $F(\dot{x}, x) = -c_1\dot{x}/x$, or $h_1(t) = c_1$, $h_2(t) = c_2$, and $h_4(t) = c_4$, which results in $F(\dot{x}, x) = -c_1c_2\dot{x}/(c_2x + c_4)$. A special case of $h_2(t) = 0$ makes $F(t)$ to be only a function of t.

Other reductions of $F(\dot{x}, x, t)$ are also possible but since the forcing function originates exclusively from the gauge function $\Phi_{gn}(x, t)$, there are other terms resulting from $L_{ns}(\dot{x}, x, t)$ (see Eqs 5.17 and 5.18), and these extra terms affect, in addition to $F(\dot{x}, x, t)$, the forms of functions $F_1(\dot{x}, x, t)$ and $F_2(\dot{x}, x, t)$. Both extra terms depend on the variables $\dot{x}$, x and t. Thus, the forces in the equations of motion $\ddot{x} = F_1(\dot{x}, x, t)$ and $\ddot{x} = F_2(\dot{x}, x, t)$ are always dissipative. Therefore, based on the presented results, we conclude that non-standard null Lagrangians can be used to introduce dissipative forces to dynamical systems.

5.3 Galilean Invariance of Dissipative Forces

As already demonstrated in section 3.1.2, the Galilean group of the metric is composed of rotations, translations, and boosts between two inertial frames of reference. The results presented in section 3.1.2 showed that the NSL given by Eq. (3.2) is Galilean invariant but the other NSL given by Eq. (3.3) is not. Following Segovia, Vestal & Musielak (2022), Galilean invariance of forces obtained in section 5.2 is now considered.

Let (x, t) be an inertial frame moving at a constant velocity, V_0, with respect to a second inertial frame, (x', t'), and let their origins coincide at $t = t' = t_0$. Thus, there are the following transformations between the systems: $x' = x - V_0 t$ and $t' = t$. By applying these transformations to the law of inertia $\ddot{x} = 0$, it is seen that Newton's first law is Galilean invariant. However, its standard Lagrangian is not Galilean invariant (e.g., Landau & Lifschitz 1969; Lévy-Leblond 1969) and it requires a special procedure that involves standard null Lagrangians to restore its Galilean invariance (Musielak & Watson 2020a).

Let us now investigate Galilean invariance of the non-standard Lagrangian for the law of inertia (see section 3.1.2) and written here as

$$L_{ns}(\dot{x}, x, t) = \frac{1}{C_1 f^2(t)[f(t)\dot{x} - a_o x + C_2]} \ , \tag{5.21}$$

where $f(t) = a_o t + v_o$. After the Galilean transformation, this Lagrangian becomes

$$L'_{ns}[\dot{x}'(t'), x'(t'), t'] = \frac{1}{C'_1 f'^{\,2}(t')[f'(t')\dot{x}'(t) - a'_o x'(t') + C'_2 + v'_o V_0]} \ . \tag{5.22}$$

Galilean invariance of $L_{ns}(\dot{x}, x, t)$ requires that its form is the same as $L'_{ns}[\dot{x}'(t'), x'(t'), t']$. For the original and transformed Lagrangians to be of the same form in the variables $x(t)$ and $x'(t')$, the following conditions must be satisfied: (i) $f'(t') = f(t)$, which requires that $a'_o = a_o$ and $v'_o = v_o$; further, it is also required that $t' = t$, as guaranteed by the Galilean transformation; (ii) $C'_1 = C_1$ is satisfied in all inertial frames; and (iii) $C'_2 + v_o V_0 = C_2$ to be valid for all Galilean observers.

Since a_o and v_o are the integration constants for the auxiliary equation (Musielak 2021), and C_1 and C_2 are the constants of integration for the law of inertia, these constants are determined by the initial conditions to be specified for a physical problem to be solved. However, both the auxiliary equation and the law of inertia are Galilean invariant; thus, the solutions to these equations must also be the same (Galilean invariant) for all Galilean observers. The latter is equivalent to the requirement that the specified initial conditions are also the same for all Galilean observers, which validates the above conditions (i) and (ii). The condition (iii) shows that $C'_2 \neq C_2$ and that the constant C'_2 must be modified by adding another constant $v_o V_0$ to it as compared to C_2. This addition is known in advance by all Galilean observers, who by their definition already agreed on the Galilean invariance. Therefore, the non-standard Lagrangian for the law of inertia given by Eq. (3.2) is Galilean invariant, which distinguishes it from the standard Lagrangian, whose original form is not Galilean invariant (e.g., Landau & Lifschitz 1969; Lévy-Leblond 1969).

Having demonstrated the Galilean invariance of the law of inertia and its non-standard Lagrangian (see section 3.1.2), we now check the Galilean invariance of the equations of motion $\ddot{x} = F_1(\dot{x}, x, t)$ and $\ddot{x} = F_2(\dot{x}, x, t)$. It is easy to verify that neither $F_1(\dot{x}, x, t)$ nor $F_2(\dot{x}, x, t)$ is Galilean invariant because they are dissipative forces that depend explicitly on both $\dot{x}$ and x. Thus, the Galilean invariance is lost when the law of inertia is converted into the second law of dynamics.

The above results demonstrate that dissipative forces in dynamics can also be defined using non-standard null Lagrangians, which is a novel way to view forces and it significantly extends the previous work on standard null Lagrangians that were used to introduce non-dissipative forces to dynamics (Musielak & Watson 2020a,b; Musielak & Vestal et al. 2020; Vestal & Musielak 2021). The presented results also show that the non-standard Lagrangian for the law of inertia preserves its Galilean invariance, which makes it different from the standard Lagrangian, whose Galilean invariance must be fixed by using a special procedure that involves standard null Lagrangians. The presented results give novel insight into the role played by non-standard Lagrangians and non-standard null Lagrangians, which seem to be more suitable for describing damped dynamical systems, while standard Lagrangians and standard null Lagrangians seem to be more applicable to undamped dynamical systems.

It must be also pointed out that the results presented were obtained within the framework of classical mechanics, and that they can be extended to quantum fields as is shown in Chapter 6.

5.4 Relationships Between Null and Non-standard Lagrangians

The results presented in my dissertation must also be considered in the context of the recent developments in theories of null Lagrangians described by Das &

Musielak (2022, 2023). According to these authors, there is a natural mapping between the NLs and NSLs, and this mapping becomes a method to generate NSLs when NLs are known. Actually, the generated NSLs, when substituted into the E-L equation, give equations of motion by using the following condition

$$\frac{d}{dt}\left[L_{null}(\dot{x}, x, t)\right] = 0. \tag{5.23}$$

An interesting result is that any null Lagrangian $L_{null}(\dot{x}, x, t)$ gives its equation of motion, which means that each null Lagrangian derived in this dissertation has the resulting equation of motion (see section 2.3.3). As already pointed out by Das and Musielak (2022, 2023), the main advantage of Eq. (5.23) is that its form is much simpler than the E-L equation, which means that it is easier to derive an equation of motion. However, its disadvantage is that the resulting equation of motion must be in a form that obeys a special relationship among the coefficients of this equation; the relationship is unique for a given dynamical system but varies for different systems.

Indeed, I performed preliminary studies by substituting the NLs obtained in this dissertation into the above condition, and found out that the resulting equations of motion were different than the original ones, and that there were special relationships between the coefficients of these equations. This is consistent with the examples presented by Das & Musielak (2022, 2023), who discussed those relationships and suggested possible solutions. Nevertheless, the problem of finding a null Lagrangian whose substitution into Eq. (5.23) would result in the originally given equation of motion still remains unsolved; thus, this problem is proposed as a future activity in the field of research of null Lagrangians and gauge functions.

5.5 Chapter Summary

In this chapter, I introduced non-standard null Lagrangians and showed how the formalism developed in Chapter 4 can be extended to also include this lesser-known family of Lagrangians. I presented a variety of applications of NLs, including a way to introduce dissipative forces by way of the non-standard gauge functions, and I discussed Galilean invariance of these forces.

In section 5.1, it was shown how to construct NSNLs and their corresponding NSGFs, and then in the section following, how to use them to introduce dissipative forces to classical dynamics. The presented results significantly extended the previous work on standard null Lagrangians, wherein they were used to introduce forces to dynamics (eg. Musielak et al., 2020; Vestal & Musielak, 2021). Further, the results described in this chapter give novel insight into the role played by non-standard Lagrangians and non-standard null Lagrangians, which seem to be more suitable for describing damped dynamical systems, while standard Lagrangians and standard null Lagrangians seem to be more applicable to undamped dynamical systems.

The chapter concludes with newly discovered relationships between non-standard Lagrangians and standard and non-standard null Lagrangians, which give new insight into mathematical and physical connections between these diverse Lagrangians.

CHAPTER 6

The Role of Null Lagrangians in Quantum Mechanics

In this chapter, the role of null Lagrangians for quantum mechanical systems will be investigated and discussed. I start by describing relevant previous work and background information. Following this, I will present my work on developing a formalism for the construction of null Lagrangians for quantum fields, which, to the best of my knowledge, has not yet been done for such fields. The Galilean invariance of the Schrödinger equation will then be discussed, and a key difference in the Schrödinger equation and its Lagrangian is highlighted.

Studies of null Lagrangians and gauge functions presented in this dissertation in Chapters 4 and 5, as well as those done previously in mathematics and described in Section 2.3.3, were mainly performed for systems given by first and second-order ODEs (e.g., Krupka 1973, 1977; Olver 1983; Olver & Sivaloganatham 1988; Crampin & Saunders 2005; Krupka et al. 2010; Olver 2022; and others). As pointed out by Krupka & Musilova (1998), the problem of finding all existing null Lagrangians and their gauge functions for all ODEs of a given order is one of the most difficult problems in variational calculus. Most previous work done by mathematicians concentrated on first-order ODEs; however, all the results presented in Chapters 4 and 5 of this dissertation deal with second-order ODEs and are therefore complementary to the previous investigations. Another important difference between previous studies and the work described in this dissertation is that the latter considers exclusively systems of physical interest, whereas the mathematicians investigated general systems without making any reference to their physical meaning.

A seminal paper on defining null Lagrangians for classical fields, which are characterized by many degrees of freedom and are described by first and second-order PDEs, was published by Hojman (1983). In the following work, Krupka & Musilova (1998), Grigorge (1999), Crampin & Saunders (2005), and Thieme (2020) studied null Lagrangians in classical field theories. To the best of my knowledge, studies of null Lagrangians in quantum field theories have not yet been done, which means that the results presented in this chapter are the first of such attempts. The main goal of this chapter is to construct first null Lagrangians and their gauge functions for non-relativistic QM, and to then use these to investigate the Galilean invariance of the Schrödinger equation and its Lagrangians. It is my hope that the presented results initiate more future work in this exciting field of study of null Lagrangians and gauge functions.

6.1 Method to Construct Null Lagrangians for Quantum Fields

Despite some work done by mathematicians (e.g., Hojman 1983; Krupka & Musilova 1998; Grigorge 1999; Crampin & Saunders 2005; Thieme 2020) for classical fields, construction of null Lagrangians and their gauge functions specifically for quantum mechanical fields has not yet been done. The results presented in this section will generalize my results from Chapters 4 and 5 to quantum fields, with special applications to the Schrödinger equation of QM.

As shown in section (2.3.3), the two main characteristics of NLs are: (i) they must yield from applying the E-L operator (see Eq. 2.1) identically zero, and (ii) there must exist a scalar function (called a gauge function) whose total derivative is equal to a given NL. In other words, any NL must have these two properties. In the following, my method to construct NLs is based on these two characteristics. However, the results presented below show that the construction of NLs must be done with caution because, in addition to mathematical constraints, there are also physical constraints, which are typically ignored by mathematicians.

Let $\eta(\vec{r}, t)$ be a scalar wave function that represents a quantum field. It is easy to show that the following null Lagrangians can be constructed

$$L_{n1} = C_1 \eta \partial_t \eta \ , \tag{6.1}$$

where C_1 is an arbitrary (real or imaginary) constant and its gauge function is $\chi_1 = C_1 \eta^2/2$, and

$$L_{n2} = C_2 \eta \partial_x \eta \ . \tag{6.2}$$

with C_2 being another arbitrary constant and the gauge function given by $\chi_1 = C_2 \eta^2/2$; note that ∂_x represents the three spatial coordinates (x, y, z). Substituting these Lagrangians into the following E-L equation

$$\frac{\partial L}{\partial \eta} - \partial_t \frac{\partial L}{\partial(\partial_t \eta)} - \nabla \frac{\partial L}{\partial(\partial_x \eta)} = 0 \ , \tag{6.3}$$

it is confirmed that both L_{n1} and L_{n2} are null Lagrangians.

Since null Lagrangians vanish under the E-L operator, multiple can be added together without changing the resulting equation of motion from the E-L, as was done in Chapter 4. Let us now add Eq. (6.1) and Eq. (6.2) together,

$$L_{n3} = L_{n1} + L_{n2} \ . \tag{6.4}$$

This is allowed as $\hat{EL}(L_{n1}) = 0$ and $\hat{EL}(L_{n2}) = 0$, thus, $\hat{EL}(L_{n1} + L_{n2}) = 0$, satisfying the first condition for a null Lagrangian. The same can be done with the gauge functions, which are known. However, the addition of gauge functions causes a problem with physical units, which must be fixed before the results are applied to physical systems. In other words, the addition of L_{n1} and L_{n2} to produce L_{n3} is not equivalent to the addition of their corresponding gauge functions $\chi_3 = \chi_1 + \chi_2$

since the constants C_1 and C_2 have different units. This illustrates the importance of looking at these systems from a physical perspective, which is required to fix this physical unit problem.

With this in mind, taking a closer look at Eq. (6.4), which becomes

$$L_{n3} = C_1 \eta \partial_t \eta + C_2 \eta \partial_x \eta \ , \tag{6.5}$$

it is apparent that this Lagrangian is also physically inconsistent due to the lack of agreement in the units. Let us now refine our approach so as to find a method to construct physically consistent null Lagrangians and their gauge functions, along with the E-L equation for the systems.

To fix the units in Eq. (6.5), let us write,

$$L_n = c(\eta \partial_t \eta + \eta (\vec{v} \cdot \nabla) \eta) \ , \tag{6.6}$$

where $\vec{v}$ is velocity of inertial frames in Galilean relativity; note that this velocity remains constant for Galilean observers, which means that it can be used to fix the unit problem, similarly to what was done in the special theory of relativity by the speed of light. The E-L equation then becomes,

$$\frac{\partial L}{\partial \eta} - \partial_t \frac{\partial L}{\partial(\partial_t \eta)} - (\vec{v} \cdot \nabla) \frac{\partial L}{\partial(\vec{v} \cdot \nabla \eta)} = 0 \ . \tag{6.7}$$

Starting with the gauge function,

$$\chi = c \frac{1}{2} \eta^2 \ , \tag{6.8}$$

and taking the total derivative,

$$\frac{d\chi}{dt} = \frac{\partial \chi}{\partial t} + (\frac{\partial \chi}{\partial \eta})(\partial_t \eta) + (\frac{\partial \chi}{\partial \eta})(\vec{v} \cdot \nabla) \eta \ , \tag{6.9}$$

we recover Eq. (6.6), which is indeed a null Lagrangian, as can be verified using Eq. (6.7).

Further, it is possible to construct another, simpler null Lagrangian, L_{n4}, from the gauge function $\chi_4 = c\eta$. Using Eq. (6.9), the corresponding null Lagrangian is found to be

$$L_{n4} = c(\partial_t \eta + (\vec{v} \cdot \nabla)\eta) \ . \tag{6.10}$$

It can be confirmed that Eq. (6.10) is a null Lagrangian using Eq. (6.7). Thus, we have constructed two physically consistent null Lagrangians for fields, Eqs (6.6) and 6.10, along with their corresponding gauge functions.

6.2 Galilean Invariance of Schrödinger Equation and its Phase Factor

As already described in Section 3.1.2, in Galilean space-time there are two metrics, given by $ds_1^2 = dx^2 + dy^2 + dz^2$ and $ds_2^2 = dt^2$, where x, y and z are spatial coordinates and t is time. The group of all allowed rotations, translations, and boosts is known as the Galilean group of the metric (Levy-Leblond 1963, 1967); its mathematical structure is given and discussed in Section 3.1.2, and it will not be repeated here. As demonstrated by Bargmann & Wigner (1954), and then by Levy-Leblond (1967, 1969), the Lie algebra associated with the Galilean group of the metric can be extended and to become the so-called extended Galilean group, whose mathematical structure is

$$G_e \ = \ [R(3) \otimes_s B(3)] \otimes_s [T(3+1) \otimes U(1)] \tag{6.11}$$

where $R(3)$ and $B(3)$ are subgroups of rotations and boosts, respectively. In addition, $T(3+1)$ is a subgroup of translations in space and time and $U(1)$ is a one-parameter unitary group, and $\otimes$ is a direct product and $\otimes_s$ represents a semi-direct product. It is important to point out that the presence of the subgroup $U(1)$ in G_e guarantees that the square of the absolute value of the Schrödinger wave function remains the same for all Galilean observers, as is required by QM.

The group G_e is the universal covering group of the Galilei group of the metric G (see Section 3.1.2), and it is used to demonstrate Galilean invariance of the Schrödinger equation (e.g., Merzbacher 1998; van Oosten 2006; Musielak & Fry 2009; Fry & Musielak 2010). As those authors showed, the Schrödinger equation is invariant with respect to all Galilean rotations and translations. However, the requirement that the Schrödinger equation be invariant with respect to Galilean boosts involves a phase factor, which is now derived.

In Galilean space-time, a boost is defined by the change of coordinate

$$\vec{r}\,' = \vec{r} - \vec{v}t \quad \text{and} \quad t' = t \ , \tag{6.12}$$

which gives

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \quad \text{and} \quad \nabla' = \nabla \ . \tag{6.13}$$

Let us now apply these Galilean boost transformations to the Schrödinger equation for a free elementary particle, as given in its standard form

$$i \ \frac{\partial \psi(\vec{r}, t)}{\partial t} \ + \ \frac{\hbar}{2m} \nabla^2 \psi(\vec{r}, t) \ = \ 0 \ . \tag{6.14}$$

For this equation to be Galilean invariant, its transformed form must be given by

$$i \ \frac{\partial \psi'(\vec{r}', t')}{\partial t'} \ + \ \frac{\hbar}{2m} \nabla'^2 \psi'(\vec{r}', t') \ = \ 0 \ , \tag{6.15}$$

where $\psi(\vec{r}, t) - \phi(\vec{r}', t')\psi'(\vec{r}', t')$, with $\phi(\vec{r}', t')$ being an arbitrary function to be determined in such a way that Eqs (6.14) and (6.15) are of the same form. This requires that the following condition is satisfied

$$\left(i \frac{\partial \phi}{\partial t'} \ - i\vec{v} \cdot \nabla' \phi + \frac{\hbar}{2m} \nabla'^2 \phi \right) \psi' + \left(\frac{\hbar}{m} \nabla' \phi - i\vec{v}\phi \right) \cdot (\nabla' \psi') \ = \ 0 \ . \tag{6.16}$$

Since $\psi' \neq 0$ and $\nabla' \psi' \neq 0$ for all $\vec{r}'$ and t', the condition reduces to

$$i \frac{\partial \phi}{\partial t'} \ - i\vec{v} \cdot \nabla' \phi + \frac{\hbar}{2m} \nabla'^2 \phi \ = \ 0 \ , \tag{6.17}$$

and

$$\frac{\hbar}{m}\nabla'\phi - i\vec{v}\phi \;=\; 0 \;. \tag{6.18}$$

which can be used to determine $\phi(\vec{r}',t')$.

From Eqs (6.17) and (6.18), one finds

$$\phi(\vec{r}\,',t') = \phi_0 \; e^{im\left(\vec{v}\cdot\vec{r}\,'+\frac{1}{2}v^2t'\right)/\hbar} \;, \tag{6.19}$$

where ϕ_0 is an integration constant. The existence of the phase function $\phi(\vec{r}\,',t')$ is an important result because it demonstrates that Schrödinger equation is Galilean invariant and that the explicit form of the transformation law for the state functions $\psi(\vec{r},t)$ and $\psi'(\vec{r}\,',t')$ can be derived. Taking $\phi_0 = 1$, the transformation law becomes

$$\psi(\vec{r},t) \;=\; \psi(\vec{r}\,'+\vec{v}t',t') \;=\; \psi'(\vec{r}\,',t')\; e^{im\left(\vec{v}\cdot\vec{r}\,'+\frac{1}{2}v^2t'\right)/\hbar} \;. \tag{6.20}$$

The obtained result is well-known and it can be found in advanced QM textbooks (e.g., Merzbacher 1998); this result was also used by different authors in their studies of the origin of the Schrödinger equation from the point of view of group theory (e.g., van Oosten 2006; Musielak & Fry 2009; Fry & Musielak 2010).

The presence of the phase factor in the transformed wave function implies that the wave functions for different Galilean observers have different forms because $\phi(\vec{r}\,',t')$ changes from one inertial frame to another. Notably, this does not violate the Galilean invariance of the Schrödinger equation as QM only requires that $|\psi(\vec{r},t)|^2 = |\psi'(\vec{r}\,',t')|^2$, since it is consistent with the presence of the subgroup $U(1)$ in G_e and it is the basis for all measurements in QM.

6.3 Galilean Invariance of the Schrödinger Lagrangian

As shown in the previous section, the Schrödinger equation is Galilean invariant and this invariance requires a phase function, whose explicit form is given by

Eq. (6.20). Recall from section 3.1.2 that while Newton's law of inertia is Galilean invariant, its Lagrangian is not (e.g., Landau & Lifschitz 1969; Lévy-Leblond 1969); however, its Galilean invariance can be restored, as shown by Musielak & Watson (2020a).

Now, I will investigate the Galilean invariance of the Schrödinger Lagrangian, which can be written as (e.g., Daughty 1990; Merzbacher 1998)

$$L_s(\psi, \psi^*, \partial_t\psi, \partial_t\psi^*, \nabla\psi, \nabla\psi^*) = -\frac{\hbar}{2m}(\nabla\psi^*) \cdot (\nabla\psi) + i\frac{1}{2}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) \ , \quad (6.21)$$

where ψ^* is the conjugate of ψ, m is mass of an elementary particle described by the equation. As most available QM textbooks do not discuss the Galilean invariance of the Schrödinger Lagrangian, in the following I will demonstrate that the Lagrangian is Galilean invariant and that this has some implications for my research, which is presented in sections 6.4 and 6.5.

Substituting $L_s(\psi, \psi^*, \partial_t\psi, \partial_t\psi^*, \nabla\psi, \nabla\psi^*)$ into the Euler-Lagrangian

$$\frac{\partial L}{\partial \psi^*} - \partial_t\left(\frac{\partial L}{\partial(\partial\psi^*)}\right) - \nabla \cdot \left(\frac{\partial L}{\partial(\nabla\psi^*)}\right) = 0 \ , \quad (6.22)$$

yields the Schrödinger equation for the wave function ψ

$$i\partial_t\psi + \frac{\hbar}{2m}\nabla^2\psi = 0 \ . \quad (6.23)$$

However, the Schrödinger equation for the conjugate wave function is obtained when ψ^* in the E-L equation is replaced by ψ (e.g., Daughty 1990).

As Galilean invariance requires that $L_s = L'_s$, where

$$L'_s(\psi', \psi'^*, \partial_t\psi', \partial_t\psi'^*, \nabla\psi', \nabla\psi'^*) = -\frac{\hbar}{2m}(\nabla'\psi'^*) \cdot (\nabla'\psi') + i\frac{1}{2}(\psi'^*\partial_{t'}\psi' - \psi'\partial_{t'}\psi'^*) \ , \quad (6.24)$$

we must now compute L'_s. To do this, let $\nabla = \nabla'$, $\hbar = const.$, $m = const.$, $\partial_{t'} = \partial_t + \vec{v} \cdot \nabla$, $\vec{r}' = \vec{r} - \vec{v}t$, $t = t'$ and

$$\psi(\vec{r}, t) = \psi(\vec{r}', \vec{t}')e^{im\vec{v}\cdot(\vec{r}'+\frac{1}{2}\vec{v}t')/\hbar} \ . \tag{6.25}$$

For simplicity of calculation, let us also define

$$\alpha = \frac{m}{\hbar}(\vec{v} \cdot \vec{r}' + \frac{1}{2}\vec{v}^2 t') \ , \tag{6.26}$$

and use

$$\psi(\vec{r}, t) = \psi'(\vec{r}', t')e^{i\alpha} \ , \tag{6.27}$$

and

$$\psi^*(\vec{r}, t) = \psi'^*(\vec{r}', t')e^{-i\alpha} \ . \tag{6.28}$$

Taking the gradient of $e^{i\alpha}$, it is found that

$$\nabla' e^{i\alpha} = i\frac{m\vec{v}}{\hbar}e^{i\alpha} \ . \tag{6.29}$$

We first find that

$$\partial_t \psi = (\partial_{t'} - \vec{v} \cdot \nabla)(\psi' e^{i\alpha}) \ , \tag{6.30}$$

which becomes

$$\psi^* \partial_t \psi = (\partial_{t'} \psi' - (\vec{v} \cdot \nabla')\psi' + i\psi' \partial_{t'} \alpha - i\psi'(\vec{v} \cdot \nabla')\alpha)e^{i\alpha} \ . \tag{6.31}$$

Using this, we then find that

$$\psi^* \partial_t \psi = \psi'^* \partial_{t'} \psi' - \psi'^*(\vec{v} \cdot \nabla')\psi' + i(\partial_{t'} \alpha - (\vec{v} \cdot \nabla')\alpha)\psi' \psi'^* \tag{6.32}$$

and the remaining terms are found similarly.

Making the substitutions and comparing the result to Eq. (6.21), we find

$$L'_s = L_s + ET_1 + ET_2 \ , \tag{6.33}$$

where ET_1 and ET_2 are the terms additional to L'_s that were not present in L_s, and are defined as

$$ET_1 = i\frac{1}{2}(\psi'(\vec{v}\cdot\nabla')\psi'^* - \psi'^*(\vec{v}\cdot\nabla')\psi') + \frac{1}{2}\frac{m}{\hbar}v^2\psi'\psi'^* \ , \tag{6.34}$$

and

$$ET_2 = -i\frac{1}{2}(\psi'(\vec{v}\cdot\nabla')\psi'^* - \psi'^*(\vec{v}\cdot\nabla')\psi') - \frac{1}{2}\frac{m}{\hbar}v^2\psi'\psi'^* \ . \tag{6.35}$$

However, looking more closely at ET_1 and ET_2, we see that the terms exactly cancel, and L_s is found to be equal to L'_s. This shows that the Schrödinger Lagrangian is indeed Galilean invariant.

6.4 Schrödinger Lagrangian and its Null Lagrangians

Since the Schrödinger equation is parabolic, its Lagrangian must be a combination of the wavefunction ψ and its conjugate, which is the only way to write Lagrangians for this type of partial differential equation (e.g., Daughty 1990). It is clear from looking at the form of the Lagrangian in Eq. (6.21) that the NLs developed in section 6.1 are not compatible with it, as they only depend on the scalar (gauge) function η and not on its conjugate. Therefore, the results presented in section 6.1 must be now generalized by also taking into account complex gauge functions and their conjugates. In the following, I describe a method to construct such NLs and their gauge functions by using the results of section 6.1.

The null Lagrangian given by Eq. (6.6) can be generalized as

$$L_{n1} = C_1[\eta(\partial_t\eta^*) + \eta^*(\partial_t\eta) + \eta(\vec{v}\cdot\nabla\eta^*) + \eta^*(\vec{v}\cdot\nabla\eta)] \ , \tag{6.36}$$

with the corresponding gauge function $\chi_1 = C_1 \eta\eta^*$, and C_1 being any (real or imaginary) constant. Since χ_1 is now a function of both η and η^*, its total derivative must be calculated by using

$$\frac{d\chi_1}{dt} = \frac{\partial \chi_1}{\partial t} + \frac{\partial \chi_1}{\partial \eta}(\partial_t \eta) + \frac{\partial \chi_1}{\partial \eta}(\vec{v}\cdot\nabla\eta) + \frac{\partial \chi_1}{\partial \eta^*}(\partial_t \eta^*) + \frac{\partial \chi_1}{\partial \eta^*}(\vec{v}\cdot\nabla\eta^*)\ . \tag{6.37}$$

Substitution of χ_1 into this equation gives the Lagrangian L_{n1}. It must be noted that χ_1 does not depend on time explicitly, but only through the time-dependence of the functions η and η^*.

Now, using the results of section 6.1, the E-L equation can be written as

$$\frac{\partial L}{\partial \eta} - \partial_t \frac{\partial L}{\partial(\partial_t \eta)} - (\vec{v}\cdot\nabla)\frac{\partial L}{\partial(\vec{v}\cdot\nabla\eta)} = 0\ , \tag{6.38}$$

with the understanding that there is the complementary E-L equation written for η^*. Substitution of $L = L_{n1}$ into these two E-L equations yields two different null Lagrangians, similar to the way in which the Schrödinger Lagrangian (see Eq. 6.21) gives two different Schrödinger equations, one for ψ and the other for ψ^* as discussed in section 6.3.

If $\eta \neq \eta(t)$, then L_{n1} reduces to L_{n2} as given by

$$L_{n2} = C_1[\eta(\vec{v}\cdot\nabla\eta^*) + \eta^*(\vec{v}\cdot\nabla\eta)]\ , \tag{6.39}$$

which is also a null Lagrangian. An interesting result is that an even simpler null Lagrangian can be constructed by taking only

$$L_{n3} = C_2[(\vec{v}\cdot\nabla\eta^*) + (\vec{v}\cdot\nabla\eta)]\ , \tag{6.40}$$

where C_2 may be a different constant than C_1 because its physical units are different.

The results obtained above can now be directly related to the Schrödinger equation by taking $\eta = \psi$ and $\eta^* = \psi^*$. Then, each of the null Lagrangians L_{n1}, L_{n2}, and

L_{n3}, or a combination of all of them, can be added to the Schrödinger Lagrangian given by Eq. (6.21); the resulting Schrödinger equation will be of the same form, which is an expected result. However, some of our previously obtained results (e.g., Musielak & Watson 2020a,b; Musielak & Vestal et al. 2020; Vestal & Musielak 2021; Vestal & Musielak 2023) demonstrated that null Lagrangians and their gauge functions can be used to restore Galilean invariance, and introduce forces and nonlinearities. In the following, I discuss some possible applications of the derived null Lagrangians and gauge functions to QM, and specifically to the Schrödinger equation.

6.5 Null Lagrangians and Schrödinger Phase Factor

The results presented in sections 6.2 and 6.3 show that both the Schrödinger equation and its Lagrangian are Galilean invariant, and that this invariance requires a phase factor. The presence of the phase factor implies that solutions to the Schrödinger equation are different in different inertial frames, which may be interpreted as a violation of Galilean invariance in QM. However, this is not the case because the Schrödinger wavefunction being a complex function in the Hilbert space cannot be directly measured experimentally; the only measurements that are allowed are those that are of the square of the absolute value of the function, which remain the same for all Galilean observers.

From a theoretical point of view, it would be interesting to explore possibility of achieving Galilean invariance for the Schrödinger equation and its Lagrangian without the phase factor but instead with help of the derived null Lagrangians and their gauge functions. Let us assume that the phase factor $\alpha = 0$, which reduces Eq. (6.34) to

$$ET_1 = i\frac{1}{2}(\psi'(\vec{v}\cdot\nabla')\psi'^* - \psi'^*(\vec{v}\cdot\nabla')\psi') \ . \tag{6.41}$$

Let us also replace $\eta = \psi'$ and $\eta^* = \psi'^*$ in Eq. (6.39) and obtain

$$L_{n2} = C_1[\psi'(\vec{v} \cdot \nabla' \psi'^*) + \psi'^*(\vec{v} \cdot \nabla \psi')] \ . \tag{6.42}$$

Comparison of these two equations shows that the terms in ET_1 are very similar to those in L_{n2} but there is a sign difference between the second term in the equations. The consequences of this sign difference are significant as it shows that the addition of L_{n2} cannot remove the extra terms resulting from the Galilean transformations. Thus, my main result of this study is that the constructed null Lagrangians cannot be used to remove the phase factor from the formulation of QM based on the Schrödinger equation.

To further explore the relationship between null Lagrangians and the Schrödinger phase factor, I assume that $\eta = e^{i\alpha}$ and $\eta^* = e^{-i\alpha}$; this is allowed as the results obtained in sections 6.1 and 6.4 are valid for any η as long as it is a scalar function. By making these substitutions to the equations for L_{n1}, L_{n2} and L_{n3} in section 6.4, it is demonstrated that $L_{n1} = L_{n2} = L_{n3} = 0$; this means that the phase factor causes all of the constructed null Lagrangians to vanish and forces their corresponding gauge functions each to simply be constants.

The main conclusion from these results is that the Schrödinger phase factor eliminates all null Lagrangians constructed for the Schrödinger equation; rather, the main terms of these null Lagrangians resemble the terms in the Schrödinger Lagrangian. Thus, the obtained results allow me to formulate (without proof) the following final conjecture.

Conjecture : *All null Lagrangians constructed for the Schrödinger wavefunction and its conjugate can be eliminated by the Schrödinger phase factor, which means that null Lagrangians cannot affect the Galilean invariance of quantum mechanics.*

However, the null Lagrangians and their gauge functions may be used to introduce different potentials to the Schrödinger equation as well as to make the equation nonlinear (e.g., Karjanto, 2019). Both topics are out of the scope of this dissertation and they will be explored as future research projects.

6.6 Summary and Future Work

While null Lagrangians have been investigated mathematically, in section 6.1 it was shown that a null Lagrangian can be mathematically consistent while also being inconsistent in its physical units. Further, a formalism for constructing null Lagrangians for fields was developed, taking into account units; this method ensures that the resulting null Lagrangians and gauge functions are physical - a necessary distinction and requirement for developing the framework of null Lagrangians for systems in physics.

In section 6.3, it was shown that the Schrödinger Lagrangian is Galilean invariant. This result is significant as the Galilean invariance of the Schrödinger equation does not guarantee the invariance of its Lagrangian, as was discussed earlier for the Law of Inertia in section 3.1.

The method in section 6.1 was generalized in section 6.4 by also taking into account complex gauge functions and their conjugates; these Lagrangians could be added to systems like the Schrödinger Lagrangian and remain physically consistent. As presented in previous chapters, the addition of a null Lagrangian can be sufficient to convert an undriven system to a driven one, introduce nonlinearities, or restore Galilean invariance (see Musielak & Watson 2020a,b; Musielak & Vestal et al. 2020; Vestal & Musielak 2021; Vestal & Musielak 2023), making this a compelling step forward for the application of null Lagrangians to systems in QM. In section 6.5, the phase factor required by the Schrödinger equation to ensure Galilean invariance was discussed and investigated. It was found that it is not possible to remove this term and replace it instead with a null Lagrangian while maintaining Galilean invariance.

The null Lagrangians developed in this chapter may well have further applications to systems in QM, just as those in prior chapters did for dynamics, and in this way they may open new doors to areas of research into quantum phenomena and shed light on underlying phenomena as of yet not understood.

CHAPTER 7

Discussion and Conclusions

The primary goal of this dissertation is to communicate the impact and potential of null Lagrangians for applications in physics, as well as the work that has been done in this area of research. As null Lagrangians have not been explored to nearly the same extent as their standard counterparts in physics, they are a natural starting point in searching for new insight into physical phenomena and underlying symmetries in our physical universe. In this dissertation, a method was presented for introducing forces to a system that is an alternative to the way in which this was done by Newton over 300 years ago. By way of the gauge functions corresponding to null Lagrangians, force arises from the physical system rather than later being added. This elegant method has been shown also to be capable of converting a linear system to one that is nonlinear.

The null Lagrangian formalism was developed first for systems in classical mechanics, with an application to the simple harmonic oscillator; it was then extended to dissipative systems. As this formalism is not confined to classical dynamics, it was also formulated for systems in quantum mechanics, building out the framework of null Lagrangians for physical systems. This work was done with physical systems in mind.

It was shown how known forces and the gauge functions corresponding to null Lagrangians are directly linked. A method to find the gauge function to introduce a given force in dynamics was presented. Further, this formalism was extended to nonlinearities, and nonlinearities and forces of key interest in physics were included as examples. A primary result of this work was to show that the addition of a null Lagrangian to a standard Lagrangian for a given system is sufficient to convert an undriven, conservative system to a driven system. Using the equations and methods developed herein to consider additional forces in physics, as well as in other fields of science, is suggested as a compelling area of future work.

The null Lagrangian formalism was then also extended to non-standard null Lagrangians, adding to knowledge of the role of null Lagrangians for physical systems and opening another area of possible future work. The development of a formalism for non-standard null Lagrangians is an additional key result of this dissertation. Gauge functions, corresponding to null Lagrangians, were also found for equations with solutions that are special functions that play significant roles in mathematical physics.

Not all null Lagrangians are physically meaningful, and, as was shown explicitly in Chapter 6, not all null Lagrangians are physically consistent; the presented results demonstrate how to restore their consistency for some systems. To the best of our knowledge, the null Lagrangians in Chapter 6 are the first null Lagrangians that have been formulated for systems in quantum mechanics; this contribution is a key result of this dissertation.

As with null Lagrangians, non-standard Lagrangians have been less of an area of focus for physical systems. However, they may also aid us in reaching a deeper understanding of physical phenomena. As was shown herein, the non-standard Lagrangian formulation of the law of inertia preserves its Galilean invariance, which is notably different from the standard Lagrangian formulation; this is another notable result of this work.

This dissertation is meant to be considered part of the starting point for what I believe to be a very promising area of research in physics. Null Lagrangians in particular have been investigated almost entirely within the field of mathematics thus far. Using this special family of Lagrangians for applications to systems in physics requires that we consider only physically consistent equations and physically meaningful applications. Rather, a mathematical treatment of these equations is not sufficient to investigate the full picture of the role that null Lagrangians play in physics. In this dissertation, I have instead approached null Lagrangians from a physical perspective.

As presented in section 4.6, the addition of a null Lagrangian is sufficient to convert an undriven system to a driven one; further, the addition of a gauge function allows for a force to arise naturally from a system without contributing to its equation of motion by way of the E-L equation. In section 4.5, it was shown that this method is capable of perfectly reproducing various known forces. As much of our physical universe is not yet understood, investigating underlying symmetries, such as the way gauge terms are able to introduce forces and nonlinearities to a system, may shed light on such phenomena. Viewing physical phenomena through this new lens may be the key to new connections and compelling discoveries. What else might we be missing by not investigating our physical world, and any underlying symmetries, further through the lens of these powerful tools?

This dissertation is not meant to fully encapsulate the role of null Lagrangians in physics. Rather, the work presented herein comprises what I hope will become the first step in a larger body of work exploring the role of null and non-standard Lagrangians for physical systems.

CHAPTER 8

Future Work

8.1 Gauge Functions and Canonical Transformations

A primary objective of this dissertation is to study gauge functions, and their corresponding null Lagrangians, within the framework of the Lagrangian formalism and its role in Classical Mechanics. As a result, canonical transformations and their generating functions, which involve Hamiltonians and Hamilton's equations, are not directly discussed in this work. Nevertheless, it is necessary to also comment on the general relationship between the gauge and generating functions, with the intention that this may set the direction for future studies.

Following Finn (2008), the extended action principle can be written as

$$\delta \int_{t_i}^{t_f} L(\dot{q}, q, \dot{p}, p, t) = 0 \ , \tag{8.1}$$

where the Lagrangian is given by

$$L(\dot{q}, q, \dot{p}, p, t) = \dot{q}p - H(q, p, t) + \frac{dF(q, p, t)}{dt} \ , \tag{8.2}$$

and with $H(q, p, t)$ being the Hamiltonian and $F(q, p, t)$ an arbitrary scalar function. It must be also noted that $\delta F(q, p, t)|_{t_i}^{t_f} = 0$.

Lagrangian dynamics allows for invertible point transformations $q = q(Q, t)$, which can be extended to become canonical transformations $(q, p, t) \rightarrow (Q, P, t)$ of phase space; these transformations preserve the form of the Hamiltonian and Hamilton's equations. Moreover, the transformations also leave the Lagrangian invariant

$$\dot{q}p - H(q, p, t) + \frac{dF(q, p, t)}{dt} = \dot{Q}P - \tilde{H}(Q, P, t) + \frac{d\tilde{F}(Q, P, t)}{dt} \ , \tag{8.3}$$

which shows that both $H(q,p,t)$ and $F(q,p,t)$ are transformed to $\tilde{H}(q,p,t)$ and $\tilde{F}(q,p,t)$, respectively. Defining $G(q,p,Q,P,t) = \tilde{F}(Q,P,t) - F(q,p,t)$, Eq. (8.3) can then be written in the following form

$$\dot{q}p - H(q,p,t) = \dot{Q}P - \tilde{H}(Q,P,t) + \frac{dG(q,p,Q,P,t)}{dt} \ , \tag{8.4}$$

where $G(q,p,Q,P,t)$ is called the generator of the canonical transformation or the generating function. In other words, a transformation $(q,p,t) \rightarrow (Q,P,t)$ is canonical if, and only if, the generating function exists.

The function $G(q,p,Q,P,t)$ is the generator of a Legendre transformation between two coordinate systems, which are connected by expressing the function by some mixture of both coordinates. To find the generating function for a given dynamical system is the main goal of the Hamilton-Jacobi equation, which is valid only if the equation of motion is completely integrable.

Comparison of the generating functions to the gauge functions derived in this dissertation shows that there are important differences between them, and that these functions play different physical roles in classical dynamics. Nevertheless, the gauge functions obtained in this dissertation may be used to derive the generating functions for a given dynamical system and for performing canonical transformations between two coordinate systems used to describe the system. This is proposed as future work in which the Lagrangian formalism studied in this dissertation is extended to the Hamiltonian formalism and its canonical transformations.

8.2 Null Lagrangians for Systems in Quantum Mechanics

Another suggested area of future work is that of further investigating the role of null Lagrangians for systems in quantum mechanics. As null Lagrangians for quantum fields were constructed in Chapter 6, they can now be considered and modified for various additional quantum systems. For systems in classical mechanics, it was shown that the addition of a null Lagrangian allows for the introduction of a force or nonlinearity, for some, but not all, null Lagrangians. It is left as an area of future research what additional physical insight can be gained from investigating null Lagrangians for systems of interest in quantum mechanics beyond those discussed in this dissertation. The null Lagrangians developed in Chapter 6 do not constitute an exhaustive list of possible null Lagrangians for systems in quantum mechanics. As such, formulating other standard and non-standard null Lagrangians for such systems is left as a supplementary area of future work.

8.3 Additional Future Work

As discussed in Chapter 5, an as-of-yet unsolved problem is that of how to find a null Lagrangian whose substitution into Eq. (5.23) would yield the originally given equation of motion. Linking the equation of motion and gauge function in this way would be a significant result and this is proposed as a future area of research in the field of null Lagrangians and gauge functions.

An additional area of suggested future research is that of finding both standard and non-standard null Lagrangians for equations of special functions of key interest in mathematical physics beyond those covered in this dissertation. Lastly, applying the formalism that was developed in Chapter 4, yielding the results presented in sections 4.5 and 4.6, to other areas of science and engineering is also suggested as future work. This method is compelling in that its applications are not limited to systems in classical dynamics; rather, is it my hope that this formalism will be used to investigate applications to systems in other fields such as theoretical biology.

APPENDIX A

In this appendix, we present a procedure for improving the bounds obtained by the application of Jensen's inequality. The methiod is based on the idea of reducing the thickness of a convex region into many thinner convex regions.

A.1 Convex Functions

A real valued function f is defined to be convex over an interval $\Omega = [\alpha, \beta]$ if

$$\lambda\Phi\{x_1) + (1-\lambda)\Phi(x_2) \geq \Phi(\lambda x_1 + (1-\lambda)x_2\}. \tag{A.1}$$

If the above inequality is reversed or

$$\lambda\Phi(x_1) + (1-\lambda)\Phi(x_2) \leq \Phi(\lambda x_1 + (1-\lambda)x_2), \tag{A.2}$$

then Φ is called concave.

A.2 Jensen's Inequality for Convex Functions

Let x be a random variable with a finite mean. If $\Phi(x)$ is real-valued convex function, then

$$E[\Phi(x)] \geq \Phi\left(E[x]\right) \tag{A.3}$$

where $E[.]$ is the mathematical expectation.

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