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FRACTAL NATURE OF THE FIBONOMIAL TRIANGLE MOD P FOR A GENERAL RANK OF APPARITION

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FRACTAL NATURE OF THE FIBONOMIAL TRIANGLE

MOD P FOR A GENERAL RANK OF APPARITION

by

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ABSTRACT

FRACTAL NATURE OF THE FIBONOMIAL TRIANGLE

MOD *P* FOR A GENERAL RANK OF APPARITION

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Pascal's Triangle forms the well-known Sierpinski Triangle fractal when divided by a prime number. The fibonomial triangle has been shown to exhibit similar behavior for certain primes. In this paper, we show that for primes p with one zero in the period of the Fibonacci sequence mod p, $\binom{n+ip^*p^m}{k+jp^*p^m}_F \equiv_p \binom{i}{j}\binom{n}{k}_F$, and for primes with two zeroes in the period, $\binom{n+ip^*p^m}{k+jp^*p^m}_F \equiv_p (-1)^{ij-nk}\binom{i}{j}\binom{n}{k}_F$. This substantially increases the size of the collection of primes for which a fractal structure is proven to exist, and the remaining case can be handled using the same methods we employ. We also describe the resulting fractals and compute their Hausdorff dimension.

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CHAPTER 1

INTRODUCTION

1.1 Binomial Coefficients and Pascal's Triangle

The binomial coefficient is defined as $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, where the factorial $n!$ is defined as $n \cdot (n-1) \ldots 2 \cdot 1$. It is conventional to define $0! = 1$ and $\binom{n}{k} = 0$ for $n < k$[4]. Binomial coefficients arise in many settings in algebra and combinatorics. One important property is that the coefficient $\binom{n}{k}$ represents the number of ways that k objects can be picked from n objects if the selection order does not matter[1].

Pascal's Triangle is a well-known structure in which the n by k'th entry is the binomial coefficient $\binom{n}{k}$. The structure of the triangle reveals an important recursive property of binomial coefficients:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Graphically, this is represented by each entry being the sum of the two entries immediately above it[1].

```
1
{1, 1}
{1, 2, 1}
{1, 3, 3, 1}
{1, 4, 6, 4, 1}
{1, 5, 10, 10, 5, 1}
{1, 6, 15, 20, 15, 6, 1}
{1, 7, 21, 35, 35, 21, 7, 1}
{1, 8, 28, 56, 70, 56, 28, 8, 1}
{1, 9, 36, 84, 126, 126, 84, 36, 9, 1}
```

Figure 1.1: Pascal's Triangle

Pascal's triangle can also be used to demonstrate an important number-theoretic property of binomial coefficients. If we divide each entry of the triangle by a prime number p, leaving its remainder, we obtain a special structure. The remainders, also called congruence classes, recur according to their appearance in base p. A number in base p is commonly written in the form $p_0 + p_1 p + \ldots p_n p^n$. It is alternatively written as $(p_n p_{n-1} \ldots p_1 p_2)_p$. The exact form of the recurrence of congruence classes is shown in the following theorem.

Theorem 1.1.0.1 (Lucas). *For a binomial coefficient* $\binom{n}{k}$, *write* n *and* k *in base* p, *obtaining* $(n_m n_{m-1} \ldots n_1 n_0)_p$ *and* $(k_m k_{m-1} \ldots k_1 k_0)_p$. *Then,*

$$\binom{n}{k} \equiv_p \binom{n_m}{k_m}\binom{n_{m-1}}{k_{m-1}} \cdots \binom{n_1}{k_1}\binom{n_0}{k_0}.$$

Note that, in the above theorem, any of the digits $k_m, \ldots k_1$ may be zero when $n > k$.

Example 1.1.1. Consider the binomial coefficient $\binom{8}{5}$. Let's assume we are interested in the coefficient's congruence class modulo 2. To apply Theorem 1.1.0.1, write $8 = 0 + 0 \cdot 2 + 0 \cdot 2^2 + 1 \cdot 2^3$ and $5 = 1 + 0 \cdot 2 + 1 \cdot 2^2 + 0 \cdot 2^3$ to obtain $(1000)_2$ and $(0101)_2$. Then

$$\binom{8}{5} \equiv_2 \binom{1}{0}\binom{0}{1}\binom{0}{0}\binom{0}{1}.$$

Since $\binom{0}{1} = 0$, we conclude that $\binom{8}{5} \equiv_2 0$, or that it is even.

If the zero entries in the triangle are colored white and the non-zero entries are colored black, the resulting object is the fractal known as Sierpinski's Gasket[5].

A characteristic property of fractals is their fractal dimension. The fractal dimension measures the complexity of the fractal's structure. Since Sierpinski's gasket can be generated iteratively, its fractal dimension can be computed in the following way: In each step, count the

```
1
{1, 1}
{1, 0, 1}
{1, 1, 1, 1}
{1, 0, 0, 0, 1}
{1, 1, 0, 0, 1, 1}
{1, 0, 1, 0, 1, 0, 1}
{1, 1, 1, 1, 1, 1, 1, 1}
{1, 0, 0, 0, 0, 0, 0, 0, 1}
{1, 1, 0, 0, 0, 0, 0, 0, 1, 1}
{1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1}
{1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1}
{1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1}
{1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1}
{1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1}
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}
{1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1}
{1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1}
{1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1}
```

Figure 1.2: Pascal's Triangle Mod 2; Zeros are Grey and Non-zeros are Bold

number N of triangles that are generated and determine the ratio r of the size of these new triangles as compared to the size of the triangles in the preceding step. Then the dimension is $\frac{log(N)}{log(r)}$[17].

Example 1.1.2. To compute the fractal dimension of the standard Sierpinski Gasket, consider the first step of its generation. The middle third of a triangle is removed, leaving three triangles in place of one. Each of these triangles is half the size of the previous one, so the fractal dimension is $\frac{log(3)}{log(2)}$.

1.2 The Fibonacci Sequence and Fibonomial Coefficients

The Fibonacci sequence is a well known sequence which appears in many unexpected contexts. It can be defined in the following way: Let $F_0 = 0$, $F_1 = 1$. Then define the rest of the sequence recursively by $F_n = F_{n-1} + F_{n-2}$.

One surprising property of the Fibonacci sequence is that the congruence classes modulo p form a periodic sequence. The length of this period is denoted by $\pi(p)$[15].

Example 1.2.1. If we divide the Fibonacci sequence by two, the first three remainders are $0, 1, 1$. After that, it is easily verifiable that the sequence repeats itself, giving $0, 1, 1, 0, \ldots$

The index of the first non-zero Fibonacci number divisible by p is called the rank of apparition and is denoted by p^*. Because the Pisano period always starts with a zero, p^* divides $\pi(p)$.

Example 1.2.2. The first four Fibonacci numbers are $0, 1, 1, 2$. We see that $F_3 = 2$, and none of the preceding Fibonacci numbers are divisible by two. For this reason, $2^* = 3$.

Unfortunately, there is not a general formula for the rank of apparition or the Pisano period which holds for all primes[12].

These properties allow the fibonomial coefficient to be defined analogously to the standard binomial coefficient. We define the fibofactorial $n!_F = F_n \cdot F_{n-1} \ldots F_2 \cdot F_1$, with $0!_F = 1$. We then define the fibonomial coefficient as

$$\binom{n}{k}_F = \frac{n!_F}{(n-k)!_F k!_F},$$

where we define $\binom{n}{k}_F = 0$ for $k > n$.

Example 1.2.3. $4!_F = F_4 \cdot F_3 \cdot F_2 \cdot F_1 = 3 \cdot 2 \cdot 1 \cdot 1 = 6$.

Example 1.2.4.

$$\binom{8}{5}_F = \frac{8!_F}{(8-5)!_F 5!_F} = \frac{F_8 \cdot F_7 \cdot F_6}{F_3 \cdot F_2 \cdot F_1} = \frac{21 \cdot 13 \cdot 8}{2 \cdot 1 \cdot 1} = 1092.$$

The fibonomial coefficient also has interesting combinatorial interpretations involving tilings of rectangles. Importantly, this can be used to show that the fibonomial coefficient is always an integer[11].

The following lemmas regarding the Fibonacci sequence and fibonomial coefficients will be useful later. They can be found in a variety of sources, including[16].

Lemma 1.2.0.1. (Lucas [9]) *For positive integers n and m,* $gcd(F_n, F_m) = F_{gcd(n,m)}$. *If* $n \mid m$, *then* $gcd(n, m) = n$, *so* $gcd(F_n, F_m) = F_n$, *and so* $F_n \mid F_m$.

Lemma 1.2.0.2. *For positive integer i and prime p,* $p \mid F_{ip^*}$.

Lemma 1.2.0.3. *For positive integers n and m,* $F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$.

The fibonomial coefficients satisfy a recurrence relation analogous to the recurrence relation for binomial coefficients:

Lemma 1.2.0.4. *For positive integers n and k,*

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F.$$

Like the binomial coefficients, the fibonomial coefficients possess a number of useful properties, among them the negation property and the iterative property:

Lemma 1.2.0.5 (Gould). *For* $n, k \in \mathbb{Z}$,

$$\binom{n}{k}_F = \binom{n}{n-k}_F.$$

The recurrence relation described in Lemma 1.2.0.4 allows one to generate the fibonomial triangle in an analagous fashion as Pascal's triangle. Computational results have suggested that the fibonomial triangle modulo p forms a fractal similar to Pascal's triangle. An example is depicted in Figure 1.3.

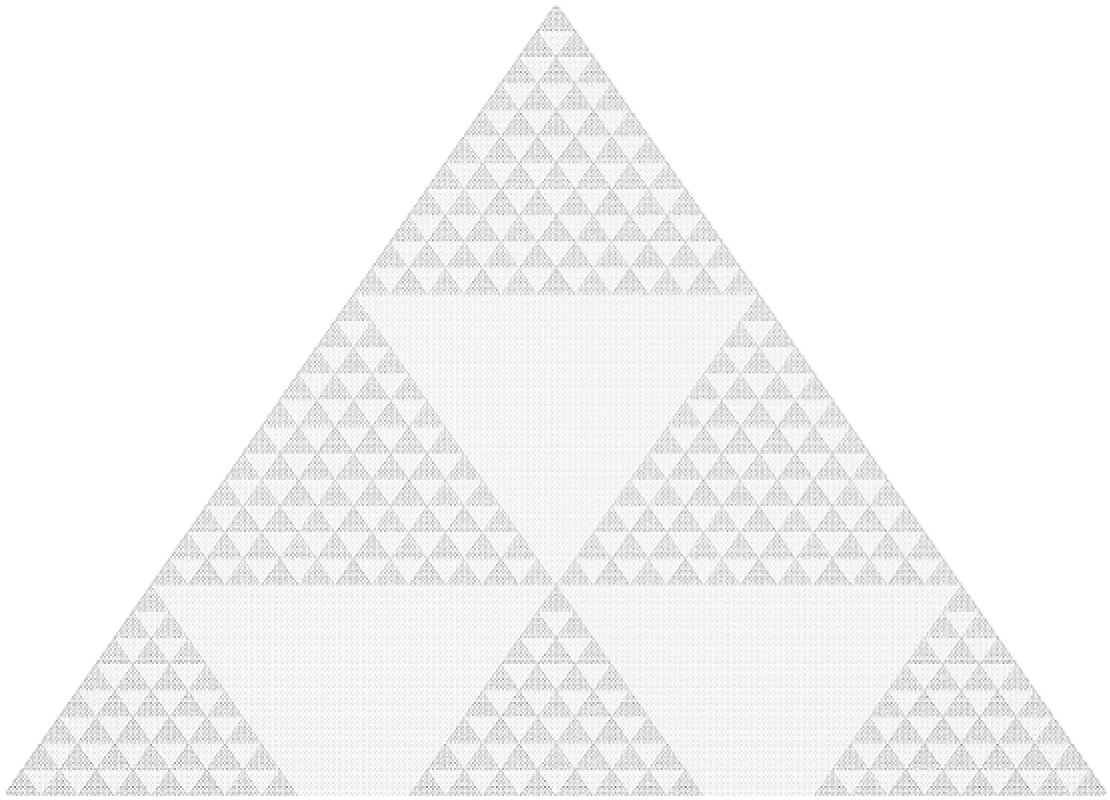

Figure 1.3: The Fibonomial Triangle Mod 11; Zeros are Grey and Non-zeros are Bold

CHAPTER 2

LITERATURE REVIEW

2.1 Modulo 2, 3, and 5 Cases

The fibonomial triangle modulo a prime was first considered for a finite number of primes. Chen and Sagan considered the modulo 2 and modulo 3 cases[2]. They provided a combinatorial, number theoretic, and inductive proof that the triangle modulo 2 forms a fractal.

Their number theoretic proof utilized Knuth and Wilf's Theorem regarding generalized binomial coefficients[8]. In order to apply this theorem, they use a special base, which will be denoted in this paper by base $\mathcal{F}_{p^*}$. An expansion of a number n in base $\mathcal{F}_{p^*}$ is of the form $n_0 + n_1 p^* + n_2 p^* p + n_3 p^* p^2 + \cdots + n_m p^* p^{m-1}$ with $n_0 < p^*$ and $n_1, n_2, \ldots, n_m < p$. It is alternatively written as $(n_m \ldots n_0)_{\mathcal{F}_{p^*}}$. Their theorem for the modulo 3 case was proven using induction. For modulo 3, they determined that the analogue of Lucass Theorem only preserved divisibility.

Theorem 2.1.0.1 (Chen and Sagan). *For $p = 2, 3$ write n and k in base $\mathcal{F}_{p^*}$. Then,*

$$p \mid \binom{n}{k}_F \iff p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F.$$

Example 2.1.1. Consider the fibonomial coefficient $\binom{8}{5}_F$. Let's assume we are interested in the coefficient's congruence class modulo 2. Observing that $2^* = 3$, to apply Theorem 2.1.0.1, we write $8 = 2 + 0 \cdot 3 + 1 \cdot 3 \cdot 2$ and $5 = 2 + 1 \cdot 3 + 0 \cdot 3 \cdot 2$ to obtain $(102)_{\mathcal{F}_{2^*}}$ and $(012)_{\mathcal{F}_{2^*}}$. Then, apply Theorem 2.1.0.1 to obtain

$$\binom{8}{5}_F \equiv_2 \binom{1}{0}_F \binom{0}{1}_F \binom{2}{2}_F.$$

Since $\binom{0}{1}_F \equiv_2 0$, we conclude that $\binom{8}{5}_F$ is divisible by 2. Recall that we computed $\binom{8}{5}_F = 1092$ in Example 1.2.4, which is indeed divisible by 2.

Southwick begins by considering fibonomial coefficients modulo 5[13]. His primary approach was number theoretic. He proved an analogue of Lucas' Theorem for for $p = 5$, and extended this to all primes p for which the rank of apparition is $p + 1$ by using Hu and Sun's Theorem on general binomial coefficients[12][7]. In the language of fibonomial coefficients, Hu and Sun's Theorem takes the following form:

Theorem 2.1.0.2 (Hu and Sun). *Let w_q be the largest divisor of F_q which is coprime to $F_1, F_2, \cdots F_{q-1}$. Then,*

$$\binom{n+iq}{k+jq}_F \equiv_{w_q} \binom{i}{j}\binom{n}{k}_F F_{q+1}^{(iq+k)(i-j)+j(n-k))}.$$

Kryuchkova and DeBellevue proved the divisibility analogue of Lucas' thereom for all primes p for which the rank of apparition is $p + 1$ through use of Knuth and Wilf's Theorem, as in[2]. They also proved the following theorem for the non-zero congruence classes:

Theorem 2.1.0.3. *For a prime p for which the rank of apparition is $p+1$, and for $0 < n < p^* p^m$, $0 \le k < p^* p^m$, $0 \le i, j < p, 0 \le m$,*

$$\binom{n+ip^*p^m}{k+jp^*p^m}_F \equiv_p (-1)^{ik-nj}\binom{i}{j}\binom{n}{k}_F.$$

CHAPTER 3

MAIN RESULTS

3.1 The Fibonomial Triangle Mod p is a Fractal

3.1.1 Necessary Lemmas

Before we begin the proof of the main theorems, the following lemmas are necessary. In all of the following, p is an odd prime.

Lemma 3.1.1.1. *If* $2p^* = \pi(p)$*, then for* $0 \leq n < p^* p^m$, $F_{n+p^*p^m} \equiv_p -F_n$.

Proof. It suffices to show that $F_{n+p^*} \equiv_p -F_n$, for then $F_{n+p^*p^m} \equiv_p (-1)^{p^m} F_n = -F_n$, as p is an odd prime.

Let $F_{1+p^*} \equiv_p v$. Then $F_{p^*} \equiv_p 0$, so $F_{2+p^*} \equiv 0 + v \equiv_p F_0 + vF_1$. Continuing in likewise fashion gives $F_{n+p^*} \equiv_p vF_n$. Then by definition of the Pisano period, $1 \equiv_p F_{1+2p^*} \equiv_p vF_{1+p^*} \equiv_p v^2$. Since the integers mod p are a field, this implies that $v \equiv_p \pm 1$. Then $v \equiv_p -1$, for otherwise, p^* would equal the period, which is a contradiction. □

Lemma 3.1.1.2. *For* $i > 0$*,*

$$\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p i(-1)^{i-1}.$$

Proof. The proof is identical to the proof in [3], where the first step follows because $2p^* = \pi(p)$ means that p^* is the semiperiod, and the semiperiod is always even [6]. □

With the above lemmas established, proof of the following two lemmas is identical to the proofs in [3].

Lemma 3.1.1.3. *For* $i > 0$,

$$\binom{ip^*p^m}{p^*p^m}_F \equiv_p i.$$

Lemma 3.1.1.4. *For* $0 \leq i, j < p$,

$$\binom{ip^*p^m}{jp^*p^m}_F \equiv_p \binom{i}{j}.$$

If the Pisano period of p is p^*, then all of the following hold.

Lemma 3.1.1.5. *For* $0 \leq n < p^*p^m$, $F_{n+p^*p^m} \equiv_p -F_n$.

Proof. Since $p^* = \pi(p)$ is the period, $F_{n+p^*} \equiv_p F_n$. Then use finite induction to obtain $F_{n+p^*p^m} \equiv_p F_n$. □

Lemma 3.1.1.6. *For* $i > 0$,

$$\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p i.$$

Proof. We prove this by induction.

First, let $i = 1$. Then the statement follows trivially.

Now, assume the inductive hypothesis:

$$\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p i.$$

Consider

$$\frac{F_{(1+i)p^*p^m}}{F_{p^*p^m}} = \frac{F_{p^*p^m+ip^*p^m}}{F_{p^*p^m}}.$$

We apply the shifting property of the Fibonacci sequence to obtain:

$$\frac{F_{p^*p^m+ip^*p^m}}{F_{p^*p^m}} = \frac{F_{p^*p^m}F_{ip^*p^m+1} + F_{p^*p^m-1}F_{ip^*p^m}}{F_{p^*p^m}}.$$

Then we simplify by cancelling like terms on the left and applying the induction hypothesis on the right, and then apply Lemma 3.1.1.5:

$$F_{ip^*p^m+1} + F_{p^*p^m-1}(i) \equiv_p 1 + i.$$

□

Lemma 3.1.1.7. *For $i > 0$,*

$$\binom{ip^*p^m}{p^*p^m}_F \equiv_p i.$$

Proof. By definition of the fibonomial coefficient,

$$\binom{ip^*p^m}{p^*p^m}_F = \frac{F_{ip^*p^m}F_{ip^*p^m-1}\dots F_{(i-1)p^*p^m}F_{(i-1)p^*p^m-1}\dots F_1}{(F_{(i-1)p^*p^m}F_{(i-1)p^*p^m-1}\dots F_1)F_{p^*p^m}F_{p^*p^m-1}\dots F_1}.$$

Cancelling like terms gives

$$\frac{F_{ip^*p^m}F_{ip^*p^m-1}\dots F_{ip^*p^m-(p^*p^m-1)}}{F_{p^*p^m}F_{p^*p^m-1}\dots F_1}.$$

The terms in the above expression take three forms, which we represent separately for clarity. Note that all reduction modulo p happens term-wise, and thus the result is an integer.

1. We first consider terms of the form $F_{ip^*p^m-a}$, where $p^* \nmid a$. For each of these terms, we identify a corresponding term in the denominator:

$$\frac{F_{ip^*p^m-a}}{F_{p^*p^m-a}}.$$

We apply Lemma 3.1.1.5 to the top so that we can cancel the top and bottom, giving just 1.

2. Next we consider terms of the form $F_{(ip^m-a)p^*}$:

$$\prod_{a=1}^{p^m-1} \frac{F_{(ip^m-a)p^*}}{F_{(p^m-a)p^*}}.$$

By Lemma 3.1.1.6,

$$\left(\prod_{a=1}^{p^m-1} \frac{F_{(ip^m-a)p^*}}{F_{(p^m-a)p^*}}\right)\left(\frac{\frac{1}{F_{p^*}}}{\frac{1}{F_{p^*}}}\right)^{p^m-1} \equiv_p \prod_{a=1}^{p^m-1} \frac{(ip^m-a)}{(p^m-a)} \equiv_p 1.$$

Note that in the modular group we use division notation to represent multiplication by an inverse.

3. The only remaining term is the quotient $\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p i$, by Lemma 3.1.1.6.

From the three cases above,

$$\binom{ip^*p^m}{p^*p^m}_F \equiv_p i.$$

□

Lemma 3.1.1.8. *For* $0 \leq i, j < p$,

$$\binom{ip^*p^m}{jp^*p^m}_F \equiv_p \binom{i}{j}.$$

Proof. Proof proceeds identically as in [3] except that Lemma 3.1.1.7 is substituted for 3.1.1.3.

□

For all cases, we'll need the following lemma:

Lemma 3.1.1.9. *For* $0 \le k < p^*p^m$, $0 \le i, j < p, 0 \le m$ $\binom{ip^*p^m}{k+jp^*p^m}_F \equiv_p 0$.

Proof. Write k in base $\mathcal{F}_{p^*}$ and then apply Theorem 2.1.0.2 to obtain

$$\begin{aligned}\binom{ip^*p^m}{k + jp^*p^m}_F &= \binom{0 + p^*(ip^m)}{k_0 + p^*(ip^m + k_1 + k_2 * p + \dots)}_F \\ &\equiv_{pr} \binom{i}{j}\binom{0}{k_1}\dots\binom{0}{k_0}_F {F_{p^*+1}}^{p(ip^m - jp^m - k_0 - k_1 \dots)}.\end{aligned}$$

Then since $k \neq 0$, there must be some l s.t. $k_l \neq 0$. If $l \ge 1$, then $\binom{0}{k_l}$ is zero by definition. If $l = 0$ then $\binom{0}{k_0}_F$ is zero by definition. In either case,

$$\binom{ip^*p^m}{k + jp^*p^m}_F \equiv_{pr} 0 \Rightarrow \binom{ip^*p^m}{k + jp^*p^m}_F \equiv_p 0.$$

□

3.1.2 Main Theorems

With necessary preliminaries dealt with, we can proceed with the main theorem.

Theorem 3.1.2.1. *For an odd prime p with pi(p) = 2p*, and for* $0 < n < p^*p^m$, $0 \le k < p^*p^m$, $0 \le i, j < p, 0 \le m$,

$$\binom{n + ip^*p^m}{k + jp^*p^m}_F \equiv_p (-1)^{ik-nj}\binom{i}{j}\binom{n}{k}_F.$$

Proof. We proceed by induction.

First let $n = k = 0$. Then the statement follows directly from Lemma 3.1.1.4.

When $n = 0$, $k > 0$, then by Lemma 3.1.1.9,

$$\binom{ip^*p^m}{k + jp^*p^m}_F \equiv_p 0 \equiv_p (-1)^{ik-0j}\binom{i}{j}\binom{0}{k}.$$

Let $n > 0$, $k \geq 0$. We assume

$$\binom{n-1+ip^*p^m}{k+jp^*p^m}_F \equiv_p (-1)^{ik-(n-1)j}\binom{i}{j}\binom{n-1}{k}_F$$

for all k.

Using the recurrence relation for fibonomial coefficients,

$$\begin{aligned}
\binom{n+ip^*p^m}{k+jp^*p^m}_F &\equiv_p F_{n+(i-j)p^*p^m-k+1}\binom{n-1+ip^*p^m}{k-1+jp^*p^m}_F + \dots \\
&\quad \dots + F_{k-1+jp^*p^m}\binom{n-1+ip^*p^m}{k+jp^*p^m}_F \\
&\equiv_p (-1)^{i-j}F_{n-k+1}(-1)^{i(k-1)-(n-1)j}\binom{i}{j}\binom{n-1}{k-1}_F + \dots \\
&\quad \dots + (-1)^j F_{k-1}(-1)^{i(k)-(n-1)j}\binom{i}{j}\binom{n-1}{k}_F \\
&\equiv_p (-1)^{ik-nj}\binom{i}{j}\left[F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F\right] \\
&\equiv_p (-1)^{ik-nj}\binom{i}{j}\binom{n}{k}_F.
\end{aligned}$$

This completes the proof. □

Theorem 3.1.2.2. *For an odd prime p with p^* even and $\pi(p) = p^*$, and for $0 < n < p^*p^m$, $0 \leq k < p^*p^m$, $0 \leq i, j < p, 0 \leq m$,*

$$\binom{n+ip^*p^m}{k+jp^*p^m}_F \equiv_p \binom{i}{j}\binom{n}{k}_F.$$

Proof. We proceed by induction.

First let $n = k = 0$. Then the statement follows directly from Lemma 3.1.1.8.

When $n = 0$, $k > 0$, then by Lemma 3.1.1.9,

$$\binom{ip^*p^m}{k + jp^*p^m}_F \equiv_p 0 \equiv_p \binom{i}{j}\binom{0}{k}.$$

Let $n > 0$, $k \geq 0$. We assume

$$\binom{n-1+ip^*p^m}{k+jp^*p^m}_F \equiv_p \binom{i}{j}\binom{n-1}{k}_F$$

for all k.

Using the recurrence relation for fibonomial coefficients,

$$\begin{aligned}
\binom{n+ip^*p^m}{k+jp^*p^m}_F &\equiv_p F_{n+(i-j)p^*p^m-k+1}\binom{n-1+ip^*p^m}{k-1+jp^*p^m}_F + \dots \\
&\quad \dots + F_{k-1+jp^*p^m}\binom{n-1+ip^*p^m}{k+jp^*p^m}_F \\
&\equiv_p F_{n-k+1}\binom{i}{j}\binom{n-1}{k-1}_F + F_{k-1}\binom{i}{j}\binom{n-1}{k}_F \\
&\equiv_p \binom{i}{j}\left[F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F\right] \\
&\equiv_p \binom{i}{j}\binom{n}{k}_F.
\end{aligned}$$

This completes the proof. □

Example 3.1.1. Consider the fibonomial coefficient $\binom{39935}{26626}_F$. Suppose we are interested in its congruence class modulo 11, for which $11^* = 10$ and $\pi(11) = 10 = 11^*$, and we write $\binom{39935}{26626}_F = \binom{5+3\cdot 10\cdot 11^3}{6+2\cdot 10\cdot 11^3}_F$. Since the conditions of Theorem 3.1.2.2 are satisfied, we obtain

$$\binom{39935}{26626}_F \equiv_{11} \binom{3}{2}\binom{5}{6}_F \equiv_{11} 0$$

where the last congruence holds because $\binom{n}{k}_F$ is defined to be zero for $k > n$.

Example 3.1.2. Consider the fibonomial coefficient $\binom{97615}{406117}_F$. Suppose we are interested in its congruence class modulo 7, for which $7^* = 8$ and $\pi(7) = 16$, and we write $\binom{97615}{406117}_F = \binom{7+4\cdot 8\cdot 7^3+5\cdot 8\cdot 7^5}{5+1\cdot 8\cdot 7^3+3\cdot 8\cdot 7^5}$. Since the conditions of Theorem 3.1.2.1 are satisfied, we apply it twice to obtain

$$\binom{97615}{406117}_F \equiv_p (-1)^{5\cdot 2749-10984\cdot 3}(-1)^{4\cdot 5-8\cdot 1}\binom{5}{3}\binom{4}{1}\binom{7}{5}_F \equiv_7 -2.$$

3.2 Characterization of the Fibonomial Fractal

Theorem 3.2.0.1. *For an odd prime p such that* $2p^* = \pi(p)$ *or* $p^* = \pi(p)$*, the fractal dimension of the fibonomial triangle modulo p is* $1 + log_p(\frac{p+1}{2})$.

Proof. By Theorems 3.1.2.1 and 3.1.2.2, we can generate the fibonomial triangle mod p fractal by incrementing m. When $m = 0$, p^*-many rows are generated. Then each time m is incremented, the result is triangle of $p^* p^m$-many rows, so to shrink the triangle down to its original size, it is necessary to scale it by a factor of p. There is one new non-zero triangle generated for each i and j, $0 \leq j \leq i < p$, so the total number of non-zero triangles generated in each step is

$$\sum_{i=0}^{p-1} i + 1 = \frac{p(p+1)}{2}.$$

Thus the fractal dimension is

$$\frac{log(\frac{p(p+1)}{2})}{log(p)} = 1 + log_p(\frac{p+1}{2}).$$

Note that this is precisely the dimension of Pascal triangle-type fractals[17]. □

CHAPTER 4

FUTURE DIRECTIONS

Since there can only be one, two, or four zeros in the period[10], and each zero corresponds to a multiple of p^*, Theorems 3.1.2.2 and 3.1.2.1 deal with two out of three possible cases for the relationship between p^* and $\pi(p)$. The remaining case should be easily dealt with according to the methods above, but computation might be somewhat complicated due to the following lemma, proven by Vinson [14].

Lemma 4.0.0.1. *If* $4p^* = \pi(p)$*, then* p^* *is odd and* $F_{n+p^*} \equiv_p vF_n$*, with* $v^2 \equiv_p -1$.

The final form of Theorems 3.1.2.2 and 3.1.2.1 is remarkably similar to Theorem 2.1.0.2. They are stronger than Theroem 2.1.0.2 because they place fewer requirements on w_q: if we take $q = p^* p^m$, Theorem 2.1.0.2 would require thqt w_q be coprime to $F_1, \ldots, F_{q-1}$. Since p divides F_{p^*} by definition, p cannot be a factor of w_q, as it is not coprime to any of the terms $F_{p^*}, F_{2p^*}, \ldots, F_{p^{m-1}p^*}$.

While it may be possible to weaken the conditions of 2.1.0.2 by other methods, it would be necessary to make an argument regarding the terms in the numerator and denominator of the fibonomial which are divisible by p. If the numerator is divisible by a higher power of p than the denominator, than the coefficient will be congruent to zero. In order to obtain a non-zero congruence class, the highest power of p dividing the numerator and highest power of p dividing the denominator must be equal. This problem is solved in Lemmas 3.1.1.3 and 3.1.1.7, but there may be more elegant solutions.

REFERENCES

[1] *Binomial coefficients*. 2011. URL: http://www.encyclopediaofmath.org/index.php?title=Binomial_coefficients&oldid=18917.

[2] Xi Chen and Bruce Sagan. “The fractal nature of the Fibonomial triangle”. In: *arXiv preprint arXiv:1306.2377* (2013).

[3] Michael DeBellevue and Ekaterina Kryuchkova. *Fractal Behavior of the Fibonomial Triangle Modulo Prime p, where the Rank of Apparition of p is p+1*. Tech. rep. Carnegie Mellon University, Summer Undergraduate Applied Mathematics Institute, 2016. URL: http://www.math.cmu.edu/CNA/summer_institute/2016/projects/fractal_behavior.pdf.

[4] *Factorial*. 2011. URL: https://www.encyclopediaofmath.org//index.php?title=Factorial&oldid=18764.

[5] Richard Guy. “The Strong Law of Small Numbers”. In: *Math Magazine* 63 (1990), pp. 3–20.

[6] Tom Harris. “Notes on the Pisano semiperiod”. In: (2016). Accessed: 2016-07-19.

[7] Hong Hu and Zhi-Wei Sun. “An Extension of Lucas Theorem”. In: *Proceedings of the American Mathematical Society* 129.12 (2001), pp. 3471–3478.

[8] Donald E Knuth and Herbert S Wilf. “The power of a prime that divides a generalized binomial coefficient”. In: *J. reine angew. Math* 396 (1989), pp. 212–219.

[9] Edouard Lucas. “Théorie des fonctions numériques simplement périodiques”. In: *American Journal of Mathematics* 1.4 (1878), pp. 289–321.

[10] Marc Renault. “The Fibonacci Sequence Under Various Moduli”. MA thesis. Wake Forrest University, 1996, p. 37.

[11] Bruce E Sagan and Carla D Savage. “Combinatorial interpretations of binomial coefficient analogues related to Lucas sequences”. In: *Integers* 10.6 (2010), pp. 697–703.

[12] Jeremiah Southwick. “A Conjecture concerning the Fibonomial Triangle”. In: *arXiv preprint arXiv:1604.04775* (2016).

[13] Jeremiah T Southwick. “Divisibility conditions for fibonomial coefficients”. PhD thesis. Wake Forest University, 2016.

[14] John Vinson. “The relation of the period modulo m to the rank of apparition of m in the Fibonacci sequence”. In: *Fibonacci Quart* 1.2 (1963), pp. 37–45.

[15] Eric W Weisstein. *Pisano Period*. URL: http://mathworld.wolfram.com/PisanoPeriod.html.

[16] Howard J Wilcox. “Fibonacci sequences of period n in groups”. In: *Fibonacci Quart* 24.4 (1986), pp. 356–361.

[17] Stephen Wolfram. “Geometry of Binomial Coefficients”. In: *American Mathematical Monthly* 91.9 (1984), pp. 566–571.

BIOGRAPHICAL INFORMATION

Michael DeBellevue is pursuing the completion of dual degrees in Mathematical Biology and Music Composition. As an undergraduate, he received the Honors Distinction Scholarship, which allowed him to freely pursue many academic opportunities.

He began his freshman year by participating in the Undergraduate Training in Theoretical Ecology Research program. As part of this program, he worked on a research project on tuberculosis disease modeling. This research project was continued with the Honors College through an Undergraduate Research Fellowship.

The summer of his junior year, he attended the Summer Undergraduate Applied Mathematics Institute at Carnegie Mellon University, where he completed a preliminary project on fibonomial coefficients. In fall he attended the prestigious Budapest Semesters in Mathematics study abroad program, where he obtained the Honors designation for academic accomplishment. His mathematics achievements were twice recognized by the University of Texas at Arlington Mathematics Department by the bestowal of the outstanding mathematics senior award.

In music, he actively participated in a number of ensembles, including the university orchestra and wind symphony. In his final semester he has begun participating in the university jazz program by playing in a jazz combo.

In music performance and composition, he received the Centennial Music Scholarship and the Lloyd Carr Taliaferro Memorial Scholarship awards. In his final semester he completed a recital of a number of his compositions for a variety of ensembles.