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ON THE VANISHING OF SELF TOR

Tatheer Ajani

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by

TATHEER AJANI

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ABSTRACT

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Tatheer Ajani, B.S. Mathematics

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The Tor functor plays a large role in homological algebra and its uses include defining generalized algebraic structures such as the homology of groups and associative algebras. Many proven properties of Tor involve two distinct R-modules A and B, but self Tor concerns a single module, in other words, $A = B$. This work takes a look at the classical case of the vanishing of Tor when A is the quotient of R by an ideal and uses the definitions and theorems we develop in order to generalize the classical case. We begin definitions of chain complexes, homology and exactness, projective modules and resolutions, syzygy, tensor products, and functors. We use that information to construct and define the Tor functor, the key element to our classical case. The main result of this work uses the isomorphism between self Tor on the $(p-1)$th syzygy module and the 1st homology module of a free resolution tensored with itself. Then using the minimal generator of the pth syzygy module, we find a 1-cycle of the free resolution tensored with itself, showing that self Tor of a finitely generated module is not 0, up to isomorphism. We conclude by demonstrating that the classical case is merely a corollary of this theorem.

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CHAPTER 1

INTRODUCTION

The derived functor Tor and its applications play a major role in homological algebra and ring theory. Used to define invariants of algebraic structures, such as homology of groups and associative algebras, we immediately see the importance of this functor by simply taking a look at its prominence in any homological algebra textbook. In this thesis, we investigate a long-standing conjecture that for a commutative ring R and a non-zero R-module M, if $\operatorname{Tor}_i^R(M,M) \neq 0$ for $i > 0$, then M is projective.

Before we delve further into this problem, we begin with important terminology, theorems, and results that are needed to understand properly the topic at hand, then we will move forward to prove our claim.

CHAPTER 2

BACKGROUND

We use this section to define any terms or concepts needed to understand later sections. An experienced reader may skim over the concepts they find familiar. However, we also intend to use this section to establish notation that will be used throughout the paper, so skipping this section entirely is not recommended.

2.1 Chain Complexes

A **chain complex *C* of *R*-modules** is a family $\{C_i\}_{i\in\mathbb{Z}}$ of R-modules together with R-module homomorphisms $\{d_i : C_i \to C_{i-1}\}_{i\in\mathbb{Z}}$ such that the composite $d_i d_{i+1} = 0$, where the maps d_i are **boundary operators**. In other words, Im $d_{i+1} \subseteq$ ker d_i for all $i \in \mathbb{Z}$. We can write out a chain complex as

$$\cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \cdots$$

We often call such a chain complex a *sequence of homomorphisms*. For our purposes, we look at chain complexes such that $C_i = 0$ for all $i < 0$, i.e.,

$$\cdots \longrightarrow C_{i+1} \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

Here, we have removed the names of the maps as they will be understood. The ***n*th homology module** is the quotient group $H_n(C) = \text{ker } d_n \,/\, \text{im } d_{n+1}$, where we call the kernel of d_n the module of ***n*-cycles of *C*** and the image d_{n+1} the module of ***n*-boundaries of *C***. This is sometimes referred to as the *homology at the nth position*.

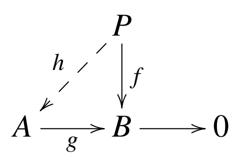
If $H_n(C) = 0$, meaning ker $d_n =$ im d_{n+1}, we say that the sequence is *exact at n*. Similarly, if

$H_n(C) = 0$ for all $n \in \mathbb{N}$, we call this an *exact sequence*. By this, we mean the complex is *exact* or has *no homology*.

2.2 Projective Resolutions

To begin the topic of projective resolutions, we must look at a special type of R-modules namely, the projective modules. Many of these definitions can be easily found in Rotman [1] and Weibel [2].

A **projective module** P is an R-module if for any R-modules, A and B, with homomorphisms $f : P \to B$ and $g : A \to B$, g a surjection, then there exists a map $h : P \to A$ such that $gh = f$, that is, the map h makes the following diagram commute, where the bottom line is exact:



Now let C be a projective R-module. A **projective resolution** of M is an exact chain complex with C_i projective for each i. Notice that for the sequence to be exact, the map d_0 must be surjective.

$$\cdots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} M \longrightarrow 0$$

We call the R-module im d_n the ***n*th syzygy module** of M.

Recall that a commutative **local ring** is a commutative Noetherian ring with a unique maximal ideal $\mathfrak{m}$. When R is a local ring, we say that the above projective resolution of M is **minimal** if im $d_n \subseteq \mathfrak{m}C_{n-1}$ for all $n \geq 1$. We note that when R is local, projective modules are necessarily free.

2.3 Tensor Products

In most cases, the reader has some familiarity with tensor products, either regarding vector spaces or abelian groups. Here, we will give a general definition of tensor product for two

R-modules. A tensor product can be viewed as a solution to a universal mapping problem. It creates a unique map that makes many diagrams commute.

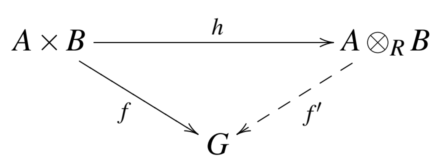
Let R be a ring, let A_R be a right R-module, let ${}_RB$ be a left R-module, and let G be an (additive) abelian group. A function $f : A \times B \to G$ is called ***R*-biadditive** if, for all $a, a' \in A, b, b' \in B$, and $r \in R$, we have

$$\begin{aligned} f(a+a',b) &= f(a,b)+f(a',b), \\ f(a,b+b') &= f(a,b)+f(a,b'), \\ f(ar,b) &= f(a,rb). \end{aligned}$$

If R is commutative and A, B, and M are R-modules, then f is called ***R*-bilinear** if f is R-biadditive and

$$f(ar,b) = f(a,rb) = rf(a,b).$$

Given a ring R and modules A_R and ${}_RB$, then their **tensor product** is an abelian group $A \otimes_R B$ where an R-biadditive function $h : A \times B \to A \otimes_R B$ such that for every abelian group G and every R-biadditive function $f : A \times B \to G$, there exists a unique homomorphism $f' : A \otimes_R B \to G$ making the following diagram commute:



However, this does not address the existence of the tensor product.

Theorem 2.1. *If R is a ring, A_R is a right R-module, and ${}_RB$ is a left R-module, then their tensor product exists.*

Proof. Let F be the free abelian group with basis $A \times B$; that is, F is free on all ordered pairs (a,b), where $a \in A$ and $b \in B$. Define S to be the subgroup of F generated by all elements of the following three types:

$$(a, b+b') - (a,b) - (a,b'),$$

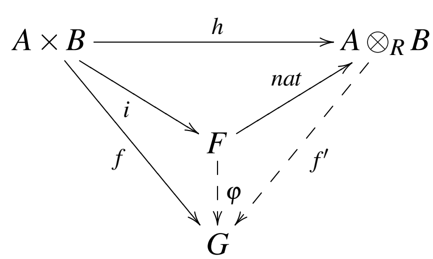
$$(a+a', b) - (a,b) - (a',b),$$

$$(ar, b) - (a, rb).$$

Define $A \otimes_R B = F/S$, denote the coset $(a,b)+S$ by $a \otimes b$, and define $h : A \times B \to A \otimes_R B$ by $h : (a,b) \mapsto a \otimes b$. We have the following identities in $A \otimes_R B$:

$$\begin{aligned} a \otimes (b+b') &= a \otimes b + a \otimes b', \\ (a+a') \otimes b &= a \otimes b + a' \otimes b, \\ ar \otimes b &= a \otimes rb. \end{aligned}$$

We can now see that h is R-biadditive. Consider the diagram:



where G is an abelian group, f is R-biadditive, and $i : A \times B \to F$ is the inclusion. Since F is free abelian with basis $A \times B$, there exists a homomorphism $\varphi : F \to G$ with $\varphi(a,b) = f(a,b)$ for all (a,b). Now $S \subseteq ker\varphi$ because f is R-biadditive, and so φ induces a map $f' : A \otimes_R B = F/S \to G$ by

$$f'(a \otimes b) = f'((a,b)+S) = \varphi(a,b) = f(a,b).$$

This equation may be rewritten as $f'h = f$; that is, the diagram commutes. □

2.4 Tor

In this section, we introduce the derived functor Tor and its properties. To begin, we will give a few definitions from category theory.

A **category** $\mathscr{C}$ is class of **objects obj($\mathscr{C}$)**, a class of sets of **morphisms** $mor(A,B)$ for each ordered pair of objects (A,B), and **composition** $mor(A,B) \times mor(B,C) \to mor(A,C)$ denoted by $(f,g) \mapsto gf$ for every triplet A,B,C of objects. We usually use $f : A \to B$ instead of $f \in mor(A,B)$. These have the following properties:

- each $f : A \to B$ has a unique domain A and a unique target B,
- for each object A, there is an identity morphism $1_A \in mor(A,A)$ such that $f1_A = f$ and $1_B f = f$ for all $f : A \to B$, and
- composition is associative, i.e., given the morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D\ ,$$

  $h(gf) = (hg)f$.

If $\mathscr{C}$ and $\mathscr{D}$ are categories, then a **functor *T***: $\mathscr{C} \to \mathscr{D}$ is a function such that

- if $A \in$ obj($\mathscr{C}$), then $T(A) \in$ obj($\mathscr{D}$),
- if $f : A \to A'$ in $\mathscr{C}$, then $T(f) : T(A) \to T(A')$ in $\mathscr{D}$, and
- if $A \xrightarrow{f} A' \xrightarrow{g} A"$ in $\mathscr{C}$, then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A")$ in $\mathscr{D}$ and $T(gf) = T(g)T(f)$.

In other words, functors are homomorphisms of categories. Building off our last definition, a **covariant functor F**: $\mathscr{C} \to \mathscr{D}$ is precisely a functor. The difference in names is to distinguish these functors from contravariant functors, which we will not discuss in this paper. With all these definitions at hand, we begin to define Tor.

Given an additive covariant functor $T : \mathscr{A} \to \mathscr{C}$ between abelian categories, where $\mathscr{A}$ has enough projective resolutions, the functors L_nT are called the **left derived functors** of T.

If B is a left R-module and $T = \Box \otimes_R B$, define $\mathrm{Tor}_n^R(\Box, B) = L_nT$. This means, for $P = \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ the chosen projective resolution of a right R-module A, then

$$Tor_n^R(A,B) = H_n(P_A \otimes_R B) = \frac{ker(d_n \otimes 1_B)}{im(d_{n+1} \otimes 1_B)}.$$

If A is a right R-module and $T = A \otimes_R \Box$, define $\mathrm{tor}_n^R(A, \Box) = L_nT$. This means, for $Q = \longrightarrow Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{\varepsilon} B \longrightarrow 0$ the chosen projective resolution of a left R-module B, then

$$tor_n^R(A,B) = H_n(A \otimes_R Q_B) = \frac{ker(1_A \otimes d_n)}{im(1_A \otimes d_{n+1})}.$$

This brings us to a nice result that will help modify our notation.

Theorem 2.2. *For all left R-modules A, all right R-modules B, and all $n \geq 0$,*

$$Tor_n^R(A,B) \cong tor_n^R(A,B)$$

For the sake of brevity, we only reference the proof of this theorem. It is an immediate consequence of Theorem 10.22 in Rotman [1], which states:

For deleted projective resolutions P_A of A and P_B of B, one has

$$H_n(P_A \otimes_R B) \cong H_n(P_A \otimes_R Q_B) \cong H_n(A \otimes_R Q_B).$$

Here, $P_A \otimes_R Q_B$ is the tensor product of complexes defined on Page 614 of Rotman's book.

CHAPTER 3

THE CLASSICAL CASE OF IDEALS

The goal of this paper is to generalize the following classical result on the vanishing of self Tor for ideals.

Theorem 3.1. *Let I be a non-zero ideal in a commutative local ring R. Then* $Tor_1^R(R/I, R/I) \neq 0$.

The usual proof of the theorem uses the classic isomorphism of Proposition 10.20 in Rotman [1], which says for ideals I and J in a commutative ring,

$$Tor_1^R(R/I, R/J) \cong (I \cap J)/IJ$$

Then when $I = J$ and R is local, $I \cap J/IJ$ is just I/I^2, which is zero iff $I = 0$, by Nakayama's Lemma.

CHAPTER 4

THE MAIN RESULT

Continuing to the final section of this paper, we can use the previously established theorems and definitions to prove our generalization.

Theorem 4.1. *Let M be a finitely generated module over a local ring R. Suppose that for a minimal free resolution F of M, the pth syzygy module has a minimal generator of the form xe_1, where $x \in R$ and e_1 is the standard basis vector of F_{p-1}. Then $Tor^R_{2p-1}(M,M) \neq 0$.*

Proof. We use the fact that

$$Tor^R_{2p-1}(M,M) \cong Tor^R_1(\text{im } d_{p-1}, \text{im } d_{p-1}) \cong H_1(F_{\geq p-1} \otimes F_{\geq p-1})$$

where $F_{\geq p-1}$ is the complex

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_p \xrightarrow{d_p} F_{p-1} \to 0$$

By hypothesis, there exists $v \in F_p$ such that $d_p(v) = xe_1$. Consider the element

$$v \otimes e_1 + (-1)^p e_1 \otimes v \in F_p \otimes_R F_{p-1} \oplus F_{p-1} \otimes_R F_p$$

We can see that $F_p \otimes_R F_{p-1} \oplus F_{p-1} \otimes_R F_p \subseteq (F \otimes_R F)_{2p-1}$.

Now, we have

$$\begin{aligned}
&(d_p \otimes 1_{F_{p-1}} + 1_{F_{p-1}} \otimes d_p)(v \otimes e_1 + (-1)^p e_1 \otimes v)\\
&= d_p(v) \otimes e_1 + (-1)^{2p-1} e_1 \otimes d_p(v)\\
&= xe_1 \otimes e_1 - e_1 \otimes xe_1\\
&= xe_1 \otimes e_1 - xe_1 \otimes e_1\\
&= 0.
\end{aligned}$$

Since xe_1 is a minimal generator of im d_p, we have

$$v \otimes e_1 \in F_p \otimes_R F_{p-1} \setminus \mathfrak{m}(F_p \otimes_R F_{p-1}),$$

and therefore have found a 1-cycle of $F_{\geq p-1} \otimes F_{\geq p-1}$. This cannot be a 1-boundary, since F was taken as minimal. Thus, we have

$$Tor^R_{2p-1}(M,M) \cong Tor^R_1(\text{im } d_{p-1}, \text{im } d_{p-1}) \neq 0.$$

□

If we recall the classical theorem of Section 3, we see that Theorem 3.1 is a simple corollary of the above theorem.

Proof. The first syzygy module of R/I is I, which certainly has minimal generators of the form xe_1. Actually, here e_1 is just 1. □

REFERENCES

[1] Joseph J. Rotman. *An Introduction to Homological Algebra*. Springer, 2009.

[2] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 2011.

BIOGRAPHICAL INFORMATION

Tatheer Ajani has attended the University of Texas at Arlington for the past three years and will graduate in May 2020 with an Honors degree in mathematics, Summa Cum Laude. During her time at UTA, she has been the vice president of the Association of Women in Mathematics, and when she was not on campus, she spent her weekends as the lead administrator at her local mosque's Saturday school. She has been accepted into the Ph.D. program at the same university and will begin graduate studies in August 2020. She is currently deciding on a specific field of study, but enjoys abstract algebra as well as number theory. She hopes to become a professor where she can combine her love of teaching and love of math.