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A STUDY OF VORTEX AND VORTICITY IN A LAMINAR FLOW

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by

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ABSTRACT

STUDY OF VORTEX AND VORTICITY

IN A LAMINAR FLOW

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Vortices are ubiquitous in nature. From kitchen sinks to galaxies, they can be found everywhere. Usually, the swirling motion of fluids comes to mind when one thinks of vortices. Being significantly important in various fields such as engineering, physics, chemistry, and aerospace, it has been extensively studied for centuries. Still, we do not have an unambiguous and universally accepted definition of a vortex. Often vorticity is used to describe the vortex, which is accurate for rigid body rotation; however, this explanation is simply not true for the fluid flow in the boundary layer. For fluid rotation, pure shear deformation needs to be considered. In order to demonstrate that, we recreated Shapiro's experiment where he used the rigid body (vorticity meter) rotation to show that the vorticity is the same thing as the vortex. Additionally, we used dyed ink to investigate if the same results still hold in fluid rotation as they did for the vorticity meter.

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CHAPTER 1

INTRODUCTION

Many vortex identification methods have been established to scrutinize vortical structure in a fluid flow. In 1858, Helmholtz put forward an idea of vorticity tube/filament to represent the vortex in the fluid flow [1]. Mathematically, it was found that the magnitude of vorticity is twice the angular speed of rotation, and the direction of vorticity is the swirling axis for a solid body. With this result, many scientists were convinced with the explanation of vortex through the concentration of vorticity and other vorticity methods [1, 2]. This result holds for a rigid body but not for fluids. However, vorticity tubes cannot constitute vortices in the turbulent viscous flow as rotation strength is minimal near the wall and shear stress is dominant. In 1991, Robinson expressed that the relation between actual vortices and firm vorticity can be rather weak [3]. People began to be skeptical towards the vorticity-based methods, which are classified as the first-generation vortex identification methods.

To overcome these inadequacies, new identification methods, such as Q, Δ, λ_2, λ_{ci}, and Ω, were introduced by different experts in this field. These methods were able to judge the presence of local rotational motion better and came with their limitations. Hunt et al. presented the Q- criterion method to visualize vortical structure more efficiently [4]. However, it is threshold-sensitive to express the area $Q > Qthreshold$ as a vortex. With the help of critical point theory, Δ- criterion was proposed by Perry and Chong, which could depict vortical structure much better [5]. Unfortunately, to visualize the iso-surface plotting effectively, we need to choose the proper threshold as this method is threshold-sensitive, too.

To deal with the downsides of the Q- and Δ- criterion, Jeong and Hussain presented the λ_2 method [6]. Still, this method is threshold-sensitive and only works well for a steady inviscid flow. The λ_{ci} vortex identification method, similar to that of the Δ- criterion, was introduced by Zhou et al. [7]. This was an improvisation of the Δ- criterion method, which visualizes the vortex structure using the imaginary part of the complex eigenvalues of the velocity gradient tensor. The limitation of this method was that it was based on the concept of the arbitrary threshold. These methods to alleviate the deficiencies of vorticity-based vortex identification are categorized as the second-generation method. The major drawback was the user-specified threshold, and the different thresholds would show distinct vortex structures. In addition, these second-generation vortex identification methods were contaminated by shears in some degrees.

As is the way of science, that is to keep pushing the boundary until the model is strictly accurate, Liu et al. published the new vortex identification method named Ω-method [8]. One of the significant advantages of this method is that it is not sensitive to the moderate threshold change. Still, all the mentioned identification methods are scalar while the fluid rotation has magnitude and direction. In 2018, Liu gave us the Liutex/Rortex method, considered one of the most significant breakthroughs in modern fluid mechanics [9]. This method represents the vortex as a vector and can give a local direction and strength of the fluid rotation. Shrestha et. al. applied all three-generations (Liutex based) vortex identification methods to Direct Numerical Simulation (DNS) data to observe the vortex structure in the flow transition. They reported that Modified Liutex-Omega method is not affected by threshold change and can show the iso-surface of vortex structure accurately [10]. Finally, Liutex-based methods provided the mathematical definition for the vortex.

CHAPTER 2

VORTEX IDENTIFICATION METHODS

2.1 First-Generation Vortex Identification Method

In the past few decades, numerous vortex identification methods have been introduced to explain vortex structures. According to Liu [11], we can classify these vortex identification methods into three generations, starting from first to third. The first is the vorticity-based method, while the second is an eigenvalue-based method. Similarly, third-generation methods are Liutex-based methods. In this chapter, we will briefly discuss all these identification methods.

2.1.1 Vorticity-based Method

The first-generation vortex identification method consists of vorticity lines, vorticity tubes, and vorticity filaments. As proposed by Helmholtz in 1858 [1], it represents the idea of vortices containing vorticity tubes, the magnitude of the vorticity gives its strength. Here, the mathematical definition of vorticity is a curl of velocity, i.e.,

$$\text{Vorticity} = \text{Curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\text{Vorticity} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \mathbf{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Helmholtz presented three theorems in fluid mechanics that explain the three-dimensional motion of fluid particles in the surrounding area of vortex filaments.

Helmholtz's first theorem: The strength of a vortex filament is constant along its length.

Helmholtz's second theorem: A vortex filament cannot end in a fluid; it must extend to the boundaries of the fluid or form a closed path.

Helmholtz's third theorem: In the absence of rotational external forces, a fluid that is initially irrotational remains irrotational [12].

2.2 Second-Generation Vortex Identification Method

Second-generation vortex identification methods are eigenvalue-based methods. These are also based on closed or spiraling streamlines. These methods were developed to undertake the limitations of the vorticity-based method. However, it comes with its own drawbacks as well. The identification methods that fall under these categories are Q criterion, Δ criterion, λ_{ci} criterion, and λ_2 criterion. We will briefly discuss each of them below.

2.2.1 Q Criterion

Given by Hunt et al. [4], it is one of the most widely used methods to visualize vortex structure. Q is defined as the residual of the vorticity tensor norm squared subtracted from the strain-rate tensor norm squared. Mathematically,

$$Q = \frac{1}{2}(\|B\|_F^2 - \|A\|_F^2)$$

Where A, B are the symmetric and antisymmetric parts of the velocity gradient tensor and $\|*\|_F^2$ represents the Frobenius norm.

$$A = \frac{1}{2}(\nabla\vec{v} + \nabla\vec{v}^T) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$B = \frac{1}{2}(\nabla\vec{v} - \nabla\vec{v}^T) = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix}$$

The region with Q > 0 can be thought as a vortex.

2.2.2 Δ Criterion

The vortex is found in the region where the velocity gradient tensor has complex eigenvalues in this method. The characteristics equation where λ_1, λ_2, and λ_3 are the eigenvalues of the 3 × 3 matrix of velocity gradient tensor can be written as

$$\lambda^3 + I_1\lambda^2 + I_2\lambda + I_3 = 0$$

Where I_1, I_2, and I_3 are the first, second and third invariants of the characteristic equation. Mathematically they are given as

$$I_1 = -(\lambda_1 + \lambda_2 + \lambda_3) = -tr(\nabla\vec{v})$$

$$I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = -\frac{1}{2}[\mathrm{tr}(\nabla\vec{v}^2) - \mathrm{tr}(\nabla\vec{v})^2]$$

$$I_3 = -\lambda_1\lambda_2\lambda_3 = -\det(\nabla\vec{v})$$

where *tr* represents trace of a matrix.

This method is a scalar method and is sensitive to the selection of iso-surface thresholds.

2.2.3 λ_{ci} Criterion

This method is a further exploration of the Δ criterion. When zero threshold is applied, this method gives the same result as that of Δ criterion. Here, the vortex structure is visualized by using the imaginary part of complex eigenvalues of the velocity gradient tensor. The tensor formation of a velocity gradient tensor can be written as

$$\nabla\vec{v} = [\vec{v}_r\ \vec{v}_{cr}\ \vec{v}_{ci}]\begin{bmatrix}\lambda_r & 0 & 0\\ 0 & \lambda_{cr} & \lambda_{ci}\\ 0 & -\lambda_{ci} & \lambda_{cr}\end{bmatrix}[\vec{v}_r\ \vec{v}_{cr}\ \vec{v}_{ci}]^{-1}$$

where λ_r is the real eigenvalue, $\vec{v}_r$is eigenvector, $\lambda_{cr} \pm i\lambda_{ci}$ are complex eigenvalues with corresponding eigenvectors $\vec{v}_{cr} \pm i\,\vec{v}_{ci}$.

2.2.4 λ_2 Criterion

This vortex identification is based on the cyclostrophic balance. This balance happens when centrifugal forces and horizontal pressure gradients push each other equally in the opposite direction. This is when we have minimal pressure on the axis of rotation. In this method, the calculation is done in a vortical region when we have minimum pressure on the axis of rotation, as mentioned above. Pressure representation in a Hessian matrix, the symmetric part S of the incompressible Navier-Stokes equation gradient can be written as:

$$S = A^2 + B^2 = -\frac{\nabla(\nabla p)}{\rho}$$

where p represents pressure.

2.3 Third-Generation Vortex Identification Method

Starting with the Liutex method, we have the Liutex-Omega method (Ω_L), Modified Liutex-Omega method ($\widetilde{\Omega}_L$), and the Liutex Core Lines method, all of which fall in third-generation vortex identification methods. The significant difference between third and other generation methods is that Liutex represents vortex as a vector while the second generation represents vector as scalar and eigenvalue related. This gives the vortex both magnitude as well as the direction of the rotation. Here, I will only be discussing the Liutex method.

2.3.1 Liutex Method

As stated above, the prominent feature of the Liutex method is that it represents the vortex as a vector. Liutex is defined as

$$\vec{R} = R\vec{r}$$

where R is the magnitude of Liutex and $\vec{r}$ is the Liutex direction.

Mathematically, Liutex represents a rigid rotation of fluids. Following the reference [13], the formula for magnitude and direction of Liutex can be given as

$$R = \vec{\omega} \cdot \vec{r} - \sqrt{(\vec{\omega} \cdot \vec{r})^2 - 4\lambda_{ci}^2}$$

where $\vec{r}$ is the real eigenvector of the velocity gradient tensor $(\overrightarrow{\nabla v})$ and λ_{ci} is the imaginary part of the conjugate complex eigenvalues of $\nabla\vec{v}$.

CHAPTER 3

VORTEX VS VORTICITY

With completion of the discussion of different generations of vortex identification methods, we have reached the part where we will discuss the difference between the vortex and vorticity and the limitation of first-generation vorticity-based vortex identification methods.

Let, $\vec{v} = (u, v, w)$ be velocity, $\overrightarrow{S_a} = (S_x, S_y, S_z)$ be angular speed and $\vec{r} = (x, y, z)$ is a location vector. Here velocity can be written as,

$$\vec{v} = \overrightarrow{S_a} \times \vec{r} = (S_x, S_y, S_z) \times (x, y, z) = (S_yz - S_zy, S_zx - S_xz, S_xy - S_yx)$$

since vorticity is a velocity curl. Mathematically,

$$\nabla \times \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (S_yz - S_zx, S_zx - S_xz, S_xy - S_yx) = (2S_x, 2S_y, 2S_z) = 2\overrightarrow{S_a}$$

This implies that vorticity is twice that of the angular speed. This way, vorticity was associated with vortex, which is true for rigid body rotation since shear is zero to negligible in this case. However, for fluid rotation, we need to consider pure shear deformation.

In simple words, vorticity can be written as,

Vorticity = Vortex + Shear

In a rigid body, with negligible shear,

Vorticity = Vortex

However, for fluid body, shear needs to be considered, so

Vorticity ≠ Vortex

That means the first-generation vortex identification method (vorticity-based) is contaminated by shear, especially in the boundary layer flow transition [14]. In order to express this, we recreated and modified Shapiro's experiment where he showed vorticity as the rotation axis and vorticity as the strength of the vortex.

3.1 Vorticity Meter and Long Channel Device

Two of the primary apparatus used in this experiment were the vorticity meter and long channel device. To observe the vorticity in a fluid flow, we designed, and 3-D printed the vorticity meter similar to the one Shapiro used in one of his experiments.

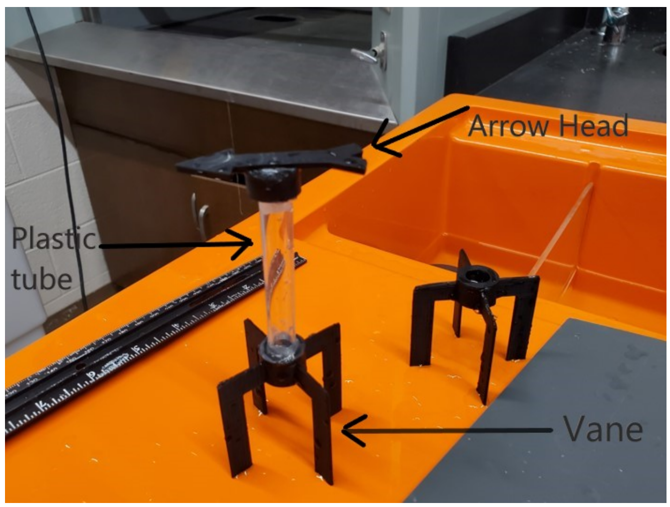


Figure 3.1: Vorticity Meter

The vanes are designed in such a way that they are at right angles and can act as paddle wheels. Vanes are attached to a plastic tube which helps the vorticity meter to float vertically. We can also change the water level inside the tube to make sure it remains afloat and does not "drown." Finally, the arrowhead gives us the direction of rotation as the vanes turn about their axis.

Another apparatus is a long channel device that consists of an open channel of rectangular cross-section supported at each end by frames. One of the frames is adjustable so that the slope of the channel can be varied. The channel walls are made from clear acrylic plastic to achieve complete visibility of the flow characteristics. This Flow Channel provides a low-cost experiment with accuracy comparable to larger-scale channel investigations. The nominal dimensions of the P6245 channel are 55 x 175 x 2500 mm (W x H x L).

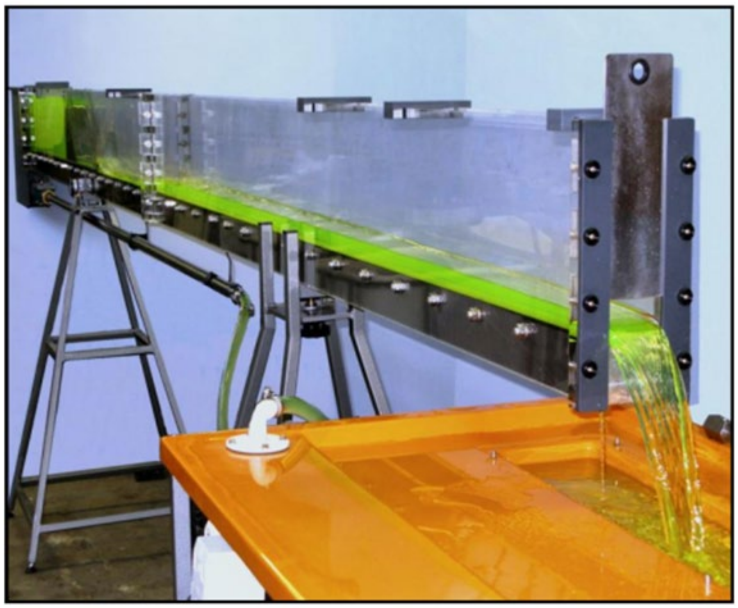


Figure 3.2: Cussons P6245 Flow channels

3.2 Reynolds Number and Laminar Flow

To determine if the flow is laminar or turbulent, we need to calculate the Reynolds number of the flow. The parameters required to achieve that are velocity and hydraulic radius of the flow, and the kinematic viscosity. This viscosity is temperature dependent, which can be determined by checking lab temperature. As the flume's width is constant, we need to determine the velocity and flow depth to calculate the Reynolds number. We used a regular ruler to determine the flow depth (ruler was attached to the flume), and velocity was determined by observing how much water the flume was discharging at a particular time. The unit we used was 'liters per minute.

Here is the complete procedure for laminar flow calculation:

1) To determine if the flow is laminar or turbulent, we need to calculate the Reynolds number of the flow. The equation for the Reynolds number is,

$$Re = \frac{VR}{\nu}$$

Where, V= Average velocity of the flow (m/s),

R = Hydraulic radius of the flow (m), and

ν = kinematic viscosity (m^2/s).

2) Velocity can be determined by observing how much water the flume was discharging in a particular time divided by the flow area.

$$V = \frac{Q}{A}$$

Where, Q = Discharge of the flow (m^3/s)

A = Area of the flow section (m^2)

3) Discharge of the flow can be determined by measuring the volume of water in a particular time.

$$Q = \frac{V}{t}$$

V = collected volume of water (*in liter*)

t = time for the water collection (*s*)

4) Hydraulic Radius can be defined by area divided by the wetted perimeter of the channel.

$$R = \frac{A}{P}$$

Where, P is the wetted perimeter of the channel

The wetted perimeter can be defined as:

$$P = W + 2 \times D$$

Where, W= Width of the channel *(m)*

D = Depth of the water *(m)*

If we measure all the parameters and insert them in equation 1, we will find the *Re* of the flow. For an open channel, if the *Re* < 500, the flow keeps laminar.

3.3 Modified Shapiro's Experiment

With all the essential things described, we are ready to dive into our experiment. First, we recreated the same experiment Shapiro did to show that vortex and vorticity are the same through rigid body rotation. The experimental setup was a long channel device with a laminar flow and vorticity meter on it. As expected, the vorticity meter rotated as it moved in the direction of the fluid flow.

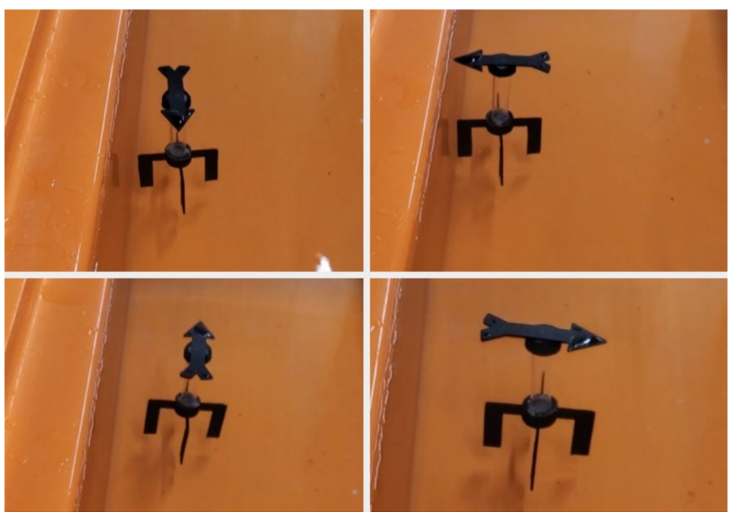


Figure 3.3: Vorticity Meter Rotation in a Laminar Flow

After that, we removed the vorticity meter and used dyed ink instead. Doing so, we observed that the dyed ink just moved in a straight line without any form of rotation.

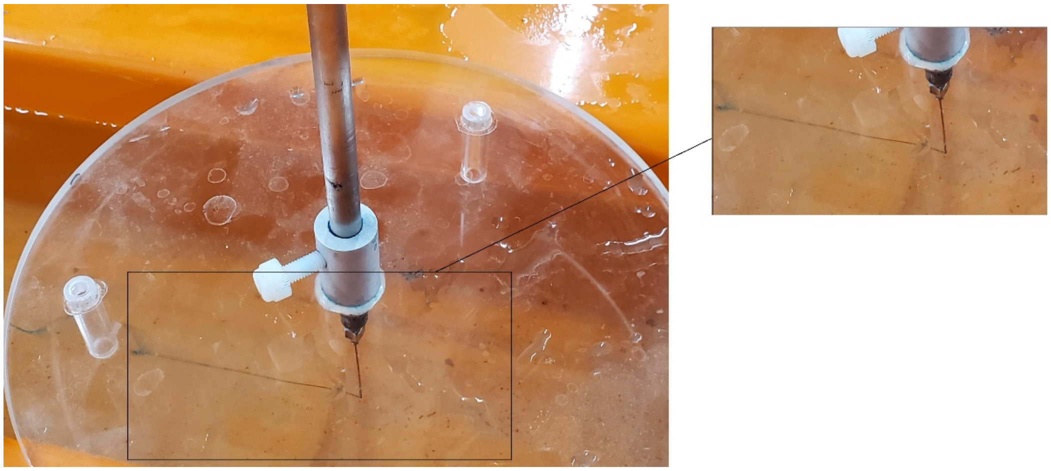


Figure 3.4: Dyed Ink Injector

The vorticity meter used in Shapiro's and our first experiment was an instrument half-submerged in water to detect the vortices formed in the water. However, the tool itself is rigid; hence it will only rotate for a rigid body where shear is negligible or zero. But the shear cannot be ignored in a real fluid. This means the fluid does not rotate, but the meter must rotate as it is a rigid body.

The major takeaway from this experiment is that even through there is no vortex in a streamline flow, the vorticity meter still rotated while the ink just moved in a straight line. This means, vorticity is not proper way to represent vortex.

3.4 Mathematical Analysis

Rewriting the derivation that shows vorticity is twice the angular speed.

Let, $\vec{v} = (u, v, w)$ be velocity, $\overrightarrow{S_a} = (S_x, S_y, S_z)$ be angular speed and $\vec{r} = (x, y, z)$ is a location vector. Here velocity can be written as,

$$\vec{v} = \overrightarrow{S_a} \times \vec{r} = \left(S_x, S_y, S_z\right) \times (x, y, z) = \left(S_y z - S_z y, S_z x - S_x z, S_x y - S_y x\right)$$

since vorticity is a velocity curl. Mathematically,

$$\nabla \times \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(S_y z - S_z x, S_z x - S_x z, S_x y - S_y x\right) = \left(2S_x, 2S_y, 2S_z\right) = 2\overrightarrow{S_a}$$

Here, the derivation works for a rigid body, but it won't for the fluid as the derivation does not take decomposition into account. We will explore this from a different aspect. Let's describe angular speed and vorticity through change of velocity, i.e., $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$.

$$\nabla \times \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (u, v, w) = (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$$

Velocity gradient tensor is a tensor made up of all derivatives of the velocity.

$$\nabla\vec{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

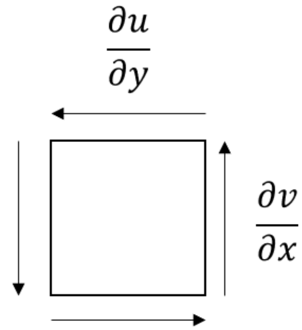
We will get, $\nabla \times \vec{v} = (\nabla\vec{v})_{32} - (\nabla\vec{v})_{23}, (\nabla\vec{v})_{13} - (\nabla\vec{v})_{31}, (\nabla\vec{v})_{21} - (\nabla\vec{v})_{12}$

$$\text{or}, \nabla \times \vec{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

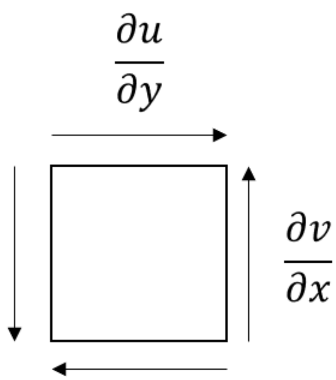
Suppose the angular speed for the x-axis and y-axis are zero, and there is only angular speed along the z-axis. In that case, the first two terms will be equal to zero, and we will have

$$\nabla \times \vec{v} = (\nabla\vec{v})_{21} - (\nabla\vec{v})_{12} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

There can be two possibilities here.



Here, the object is rotating in an anti-clockwise direction as both $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ are in an anti-clockwise direction.



Next, $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ have the same magnitude. However, their direction is in opposite direction as $\frac{\partial v}{\partial x}$ is moving anti-clockwise while $\frac{\partial u}{\partial y}$ is moving in a clockwise direction.

For a rigid body, since there is no deformation, we have $\left|\frac{\partial u}{\partial y}\right| = \left|\frac{\partial v}{\partial x}\right|$. Suppose an object is rotating around a fixed axis with a fixed angular speed $\overrightarrow{S_a} = (0,0,S_z)$. Let the velocity be $\vec{v} = (u, v, 0)$ and the location vector $\vec{r} = (x, y, 0)$.

So,

$$\vec{v} = \overrightarrow{S_a} \times \vec{r} = (0,0,S_z) \times (x, y, 0) = (-S_z y, S_z x, 0)$$

$$\frac{\partial u}{\partial y} = \frac{\partial - S_z y}{\partial y} = -S_z$$

$$\frac{\partial v}{\partial x} = \frac{\partial S_z x}{\partial x} = S_z$$

With this, we can see that the result that implies vorticity is rotation means $\left|\frac{\partial u}{\partial y}\right| = \left|\frac{\partial v}{\partial x}\right|$ i.e., the absolute value of change in the x-component (u) of velocity with respect to y and change in the y-component (v) of velocity with respect to x are equal. The matrix to represent rigid body rotation at different axes are as follows.

Rotation matrix (rigid rotation) around the z-axis.

$$\begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rotation matrix (rigid rotation) around the y-axis.

$$\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}$$

Rotation matrix (rigid rotation) around the x-axis.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{bmatrix}$$

However, in most cases for fluid, $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$. Also, when $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$, there will be a problem in decomposing the velocity gradient tensor to a rotational matrix. The velocity gradient tensor does not only have a rotation but is coupled with rotation, shear and stretching. Hence, we use Cauchy-Stokes decomposition to decompose the velocity gradient tensor. We have,

$$\nabla\vec{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

From Cauchy-Stokes decomposition,

$$\nabla\vec{v} = A + B$$

where,

$$A = \frac{1}{2}(\nabla\vec{v} + \nabla\vec{v}^T)$$

$$B = \frac{1}{2}(\nabla\vec{v} - \nabla\vec{v}^T)$$

Now,

$$A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & -\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix}$$

Here, A and B represent a deformation and rotation matrix respectively. Also, B is the matrix that represents vorticity, which does not consider deformation at all.

3.5 Vorticity and Liutex

A local fluid rotation axis is defined as a vector that can only have stretching (compression) along its length. It is one of the fundamental properties that the rotational axis cannot be stretched or compressed or deform or rotate itself in any other direction than along its length. Also, the change in velocity of the rotational axis can only be in its rotational axis direction.

All rotations must follow this property of the rotational axis. We can write the increment of $\vec{v}$ in the direction of $d\vec{r}$ is $d\vec{v} = \nabla\vec{v} \cdot d\vec{r}$. By following the property of the rotational axis, it must satisfy that $d\vec{v} = \nabla\vec{v} \cdot d\vec{r} = \alpha d\vec{r}$ along the rotation axis, which indicates $d\vec{r}$ is the real eigenvector of $\nabla\vec{v}$.

First, let's analyze if the vorticity satisfies this fundamental concept of the rotational axis. We have,

$$d\vec{v} = \nabla\vec{v} \cdot \vec{\omega}$$

where $\vec{\omega}$ is a vorticity. From Cauchy-Stokes decomposition,

$$d\vec{v} = (A + B) \cdot \vec{\omega}$$

$$d\vec{v} = A \cdot \vec{\omega} + B \cdot \vec{\omega}$$

Assuming $\vec{\omega} = a_1\vec{r_1} + a_2\vec{r_2} + a_3\vec{r_3}$ where $\vec{r_1}, \vec{r_2}\ and\ \vec{r_3}$ and B is a rotation (vorticity) matrix, we have,

$$d\vec{v} = A \cdot (a_1\vec{r_1} + a_2\vec{r_2} + a_3\vec{r_3}) + (\nabla \times \vec{v}) \times \vec{\omega}$$

Now, $A\vec{r_1} = \lambda\vec{r_1}$,

$$d\vec{v} = a_1\lambda_1\vec{r_1} + a_2\lambda_2\vec{r_2} + a_3\lambda_3\vec{r_3} + 0$$

Unless $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

$$d\vec{v} = a_1\lambda_1\vec{r_1} + a_2\lambda_2\vec{r_2} + a_3\lambda_3\vec{r_3} + 0 \neq \lambda(a_1\vec{r_1} + a_2\vec{r_2} + a_3\vec{r_3}) = \lambda\vec{\omega}$$

From here, we can see that, in general, Vorticity is not a local fluid rotational axis.

Now, let's compare if Liutex satisfies this condition. Based on the definition of Liutex (section 2.3.3), Liutex is defined as

$$\vec{R} = R\vec{r}$$

where R is the magnitude of Liutex and $\vec{r}$ is the Liutex direction. Here, R is given as

$$R = \vec{\omega} \cdot \vec{r} - \sqrt{(\vec{\omega} \cdot \vec{r})^2 - 4\lambda_{ci}^2}$$

where $\vec{r}$ is the real eigenvector of the velocity gradient tensor $(\overrightarrow{\nabla v})$ and λ_{ci} is the imaginary part of the conjugate complex eigenvalues of $\nabla\vec{v}$. Since the Liutex direction $(\overrightarrow{r})$ is an eigenvector of the velocity gradient tensor, it automatically satisfies this condition.

By finding the flaw in vorticity to represent rotation direction, we investigate the magnitude calculation from vorticity and Liutex in Couette flow.

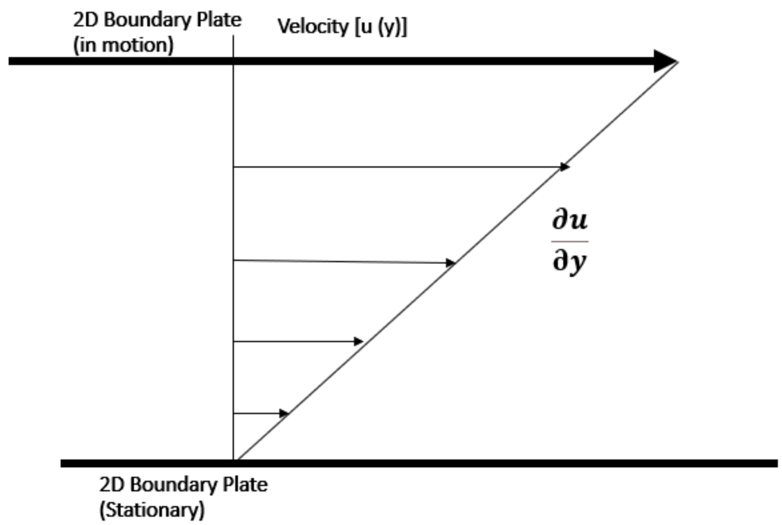


Figure 3.5: Couette Flow

Some of the assumptions made in Couette Flow are:

- Couette flow is constant pressure flow ($\frac{\partial p}{\partial y}$; the variation of pressure in y–direction is zero).
- The only nonzero velocity component is u (v = w = 0) as the flow is consistent and directed in the x-direction.

Here, the velocity gradient tensor takes the given form.

$$\nabla\vec{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}; \text{ where a is some magnitude.}$$

Now, calculating the magnitude of vorticity in a Couette flow.

$$\text{Vorticity} = \nabla \times \vec{v} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix} = (0, 0, -a).$$

Even though there is no rotation in the Couette flow, the vorticity is still giving us some magnitude value. This is the same as our experiment result where the vorticity meter rotated even when there is no rotation in the laminar (streamline) flow.

With this, we will calculate the magnitude of Liutex in a Couette flow. The magnitude of Liutex is given as follows:

$$R = \vec{\omega} \cdot \vec{r} - \sqrt{(\vec{\omega} \cdot \vec{r})^2 - 4\lambda_{ci}^2}$$

Calculating the eigenvalues of Liutex from the characteristic equation:

$$|\, A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} \lambda & -a \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \lambda = 0,0$$

which are both real eigenvalues. Hence, λ_{ci} goes to zero. Now, we have,

$$R = \vec{\omega} \cdot \vec{r} - \sqrt{(\vec{\omega} \cdot \vec{r})^2} = 0$$

Here, we have the magnitude of Liutex as zero, which is accurate for the Couette flow as there is no rotation in the system.

From these, we can conclude that Liutex is more reasonable than vorticity to investigate both magnitude and rotation of the vortex.

CHAPTER 4

CONCLUSION AND DISCUSSION

The two major misunderstandings regarding vortex come from the derivation of vorticity as an equivalent of a vortex and the uses of a solid body to show fluid rotation. In Shapiro's experiment, he used rigid body rotation to show that vorticity is the same as the vortex. This is not essentially wrong for a rigid body rotation as it lacks shear. Vorticity can act as a rotation when there is negligible to zero shear, so the vorticity becomes rotation for the rigid body. However, fluid does not have the same properties as that of a rigid body. In fluid rotation, there is a pure shear deformation that needs to be considered.

Proper understanding of fundamental concepts such as vortex and vorticity in fluid mechanics can help us to expand our research extensively. This can also provide aid in turbulence research as vortices constitute a significant component of turbulent flow. Turbulence is a fluid motion characterized by chaotic changes in pressure and flow velocity. By detecting the vortex precisely, we will learn about the turbulence in the fluid flow properly. These results can be used in engineering designs to tackle the pressure and flow velocity changes in fluid motion.

In conclusion, vortex and vorticity are two different things in a fluid motion. Vorticity cannot represent vortex in a fluid flow as a pure shear deformation is present in the fluid, especially near the wall. Hence, vorticity does not always imply rotation.

APPENDIX A

LIST OF SYMBOLS

Q = Q Criterion

Δ = Delta Criterion

λ_2 = Lambda-2 Criterion

λ_{ci} = Lambda-ci Criterion

Ω = Omega Criterion

$\|*\|_F^2$ = Frobenius Norm

$\nabla\vec{v}$ = Gradient Velocity Tensor

Ω_L= Liutex-Omega Method

$\widetilde{\Omega}_L$= Modified Liutex-Omega Method

$\vec{R}$ = Liutex Vector

R = Liutex Magnitude

$\vec{r}$ = Liutex Direction

Re = Reynolds Number

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BIOGRAPHICAL INFORMATION

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Apart from academics, he likes to read books, play guitar, and listen to Bob Dylan a lot.