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BLOWING BUBBLES (AND SEEING WHERE THEY GO): A STUDY OF PARTITION SPACES AND CONTINUOUS FUNCTIONS BETWEEN THEM

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by

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ABSTRACT

BLOWING BUBBLES (AND SEEING WHERE THEY GO): A STUDY

OF PARTITION SPACES AND CONTINUOUS

FUNCTIONS BETWEEN THEM

Gabriel Cantanelli, B.S Mathematics

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Faculty Mentor: Barbara Shipman

The field of topology is concerned with the study of “topological spaces,” mathematical objects used to describe space and change at their most fundamental levels. Within this thesis, a study of a class of topological spaces, called partition spaces, is conducted. Four results concerning such spaces are presented, together with formal proofs and illustrative examples. The first result describes the behavior of limits of sequences in partition spaces. The second result characterizes continuous functions between such spaces. From the first and second results, a third finding is derived that relates continuous functions between partition spaces to limits of sequences in their domains. Lastly, the fourth result establishes necessary and sufficient conditions for a function between partition spaces to be a homeomorphism. These results are not found explicitly within the

mathematical literature and are self-contained in their development. Together, they comprise a basic description of continuous functions between partition spaces.

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CHAPTER 1

A BRIEF INTRODUCTION TO TOPOLOGY

1.1 A Pictorial Preview

This thesis begins with a quick preview in pictures of the concepts that will follow. In order to construct the primary object of study within this work, a set X is first partitioned into disjoint “bubbles”, called “prime open sets”.

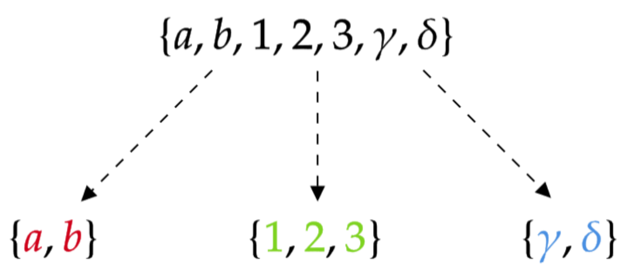

Figure 1.1: A Set X, Together With a Partition on It

Then, the collection of all possible unions of these prime open sets is formed; these unions are called “open sets”, and the collection is called a “partition topology on X”.

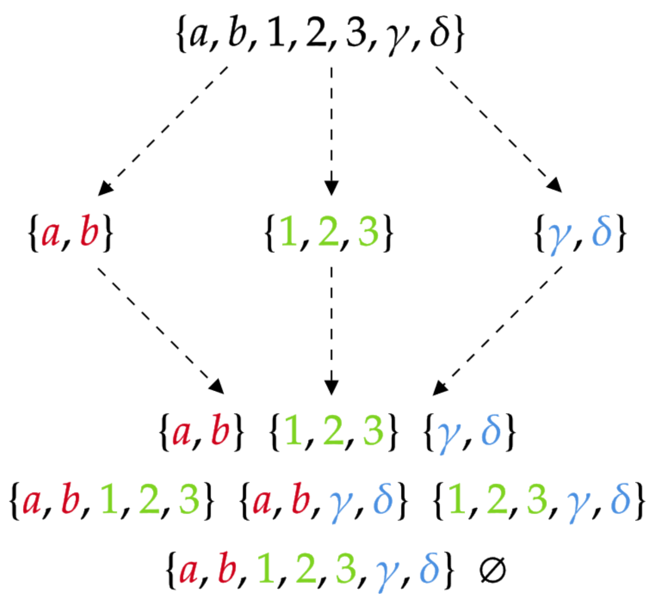

Figure 1.2: A Partition Topology on X

The set X, together with the partition topology on it, is called a “partition space”; this is the primary object of interest.

It will then be shown that the behavior of continuous functions between partition spaces is completely characterized by the “preservation” of prime open sets in the domain,

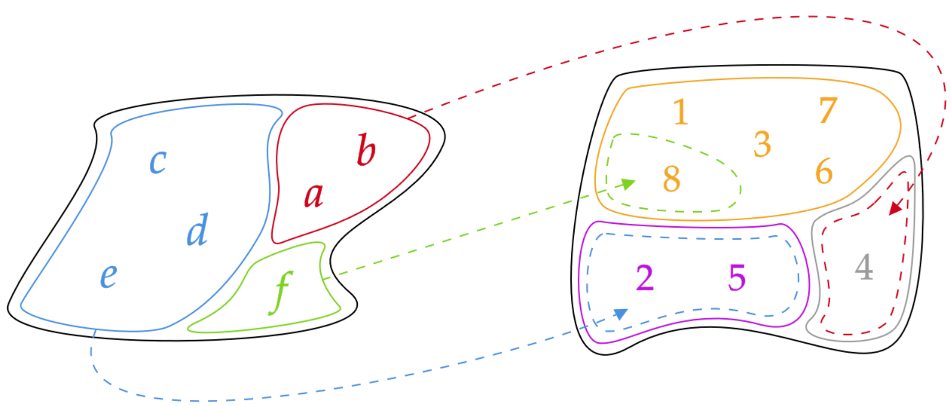

Figure 1.3: A Continuous Function Between Partition Spaces, Which Keeps the Bubbles Intact!

and that the behavior of non-continuous functions between such spaces is characterized by the "breaking" of prime open sets in the domain.

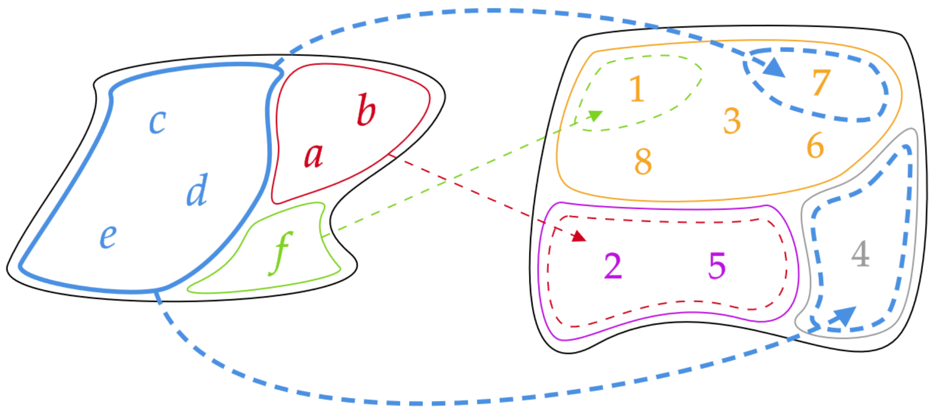

Figure 1.4: A Non-Continuous Function Between Partition Spaces, Which "Pops" the Bubbles!

This result serves as the foundation for two of the other presented results and provides the basis for the main conclusion of this work, which is that the characteristics of a partition space are completely determined by its prime open sets. Some basic definitions that will be used throughout the paper will now be discussed in order to understand the mathematics behind these concepts.

1.2 What is a Topological Space?

The first definition to understand is that of a topological space (Morris, 2020), which is the primary object of study in the field of topology.

1.2.1 Definition of a Topological Space

Definition: Let X be a set. Let T be a collection of subsets of X. We call T a "*topology on X*" if it satisfies the following four conditions:

(a). $X \in T$.

(b). $\emptyset \in T$.

(c). T is closed under arbitrary unions.

(d). T is closed under finite intersections.

The elements of T are referred to as "*open sets*". X, together with T, are referred to as a single "*topological space*", which is denoted by the pair (X, T).

Here, an arbitrary union refers to a union of any number of sets, whether the number be finite or infinite. In the same way, a finite intersection refers to an intersection of a finite number of sets. In both contexts, the sizes of the individual sets themselves (i.e., the numbers of elements that they contain) do not matter.

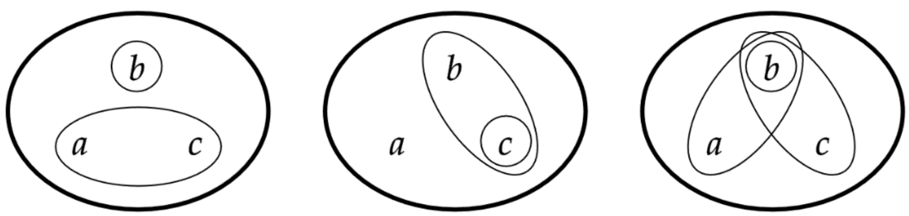

Figure 1.5: Examples of Basic Topological Spaces, in Which the "Bubbles" Represent the Open Sets

As is often the case in mathematics, this seemingly simple definition gives rise to a vast field of study. Similar to how the field of linear algebra is largely founded upon the study of vector spaces and linear maps between them, or how much of abstract algebra builds upon the basic definition of a group, the subject of topology is based upon the study of topological spaces, together with other related concepts. Particularly, topology is concerned with the various characteristics that a topological space can have, and how those characteristics interact with different types of mappings between topological spaces (University of Waterloo, 2015).

One can imagine that many kinds of topological spaces exist, each having their own unique properties and characteristics. Indeed, a whole classification system for topological spaces exists simply based upon how points in their open sets are "separated" out from one another (the "separation axioms") (Seebach & Steen, 1978). Even the "size" of a topological space can vary greatly – one can construct topological spaces using finite topologies on finite sets, finite topologies on infinite sets, or infinite topologies on infinite sets!

1.2.2 Examples of Topological Spaces

An example of a topological space (X, T) based upon a finite topology on a finite set is:

$$X = \{a, b, c, d\}$$

$$T = \big\{\emptyset, \{a, b, c, d\}, \{a, b\}\big\}$$

It is quickly verified that T satisfies all the criteria needed in order to be a topology on X. Immediately, it can be seen that $\emptyset \in T$ and $X \in T$. Verifying that T is closed under all possible unions:

$$\emptyset \cup \{a, b, c, d\} = \{a, b, c, d\} \in T$$

$$\emptyset \cup \{a, b\} = \{a, b\} \in T$$

$$\{a, b, c, d\} \cup \{a, b\} = \{a, b, c, d\} \in T$$

$$\emptyset \cup \{a, b, c, d\} \cup \{a, b\} = \{a, b, c, d\} \in T$$

It is also quickly verified that T is closed under all finite intersections (which, as T is finite, is just all possible intersections):

$$\emptyset \cap \{a, b, c, d\} = \emptyset \in T$$

$$\emptyset \cap \{a, b\} = \emptyset \in T$$

$$\{a, b, c, d\} \cap \{a, b\} = \{a, b\} \in T$$

$$\emptyset \cap \{a, b, c, d\} \cap \{a, b\} = \emptyset \in T$$

So indeed, T is a topology on X, in which case (X, T) is a genuine topological space.

An example of a topological space (Y, S) comprised of a finite topology on an infinite set is the "single point topological space", in which:

$$Y = \mathbb{R}$$

$$S = \{\mathbb{R}, \emptyset, \{x\}\}$$

Here, x denotes a real number.

An example of a topological space based upon an infinite topology on an infinite set is the topological space $(\mathbb{R}, T_{\mathbb{R}})$, where $T_{\mathbb{R}}$ is the "standard topology" on $\mathbb{R}$ formed by taking the open sets to be the usual open sets in $\mathbb{R}$ (Abbott, 2015).

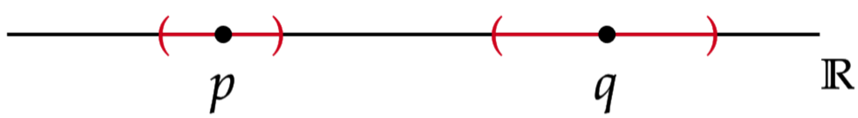

Figure 1.6: The Topological Space $(\mathbb{R}, T_{\mathbb{R}})$

Another such example is the "stalagmite topological space" $(\mathbb{N}, T_{\mathbb{N}})$ (Shipman & Stephenson, 2022), in which the topology $T_{\mathbb{N}}$ is given by:

$$T_{\mathbb{N}} = \{\{1,2, \dots, k\}: k \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$$

$(\mathbb{N}, T_{\mathbb{N}})$ can be visualized using the following figure:

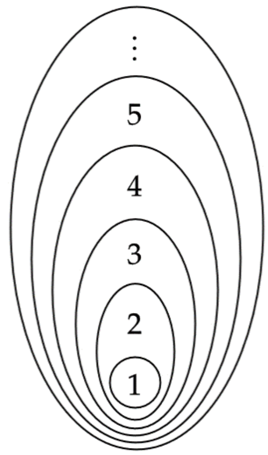

Figure 1.7: The Stalagmite Topological Space $(\mathbb{N}, T_{\mathbb{N}})$

The open sets in $T_{\mathbb{N}}$ appear to creep upwards as they encompass more natural numbers, similar to how a stalagmite creeps upwards from a cave floor.

1.3 Sequential Limits in Topological Spaces

Next presented is the definition of what it means to be a limit of a sequence in the context of topology (Shipman & Stephenson, 2022). Throughout the remainder of this work, it will be assumed that any set defined without explicit specification of its elements is nonempty.

1.3.1 Definition of a Limit of a Sequence

Definition: Let (X, T) be a topological space. Let (x_n) be a sequence in X. Let $L \in X$. To say that "L *is a limit of* (x_n)" means that all but finitely many terms of (x_n) are contained in every open set in T containing L.

Some may observe that, in the context of real analysis, this definition is equivalent to the standard definition of a sequential limit (Abbott, 2015). There, the relevant topological space is $(\mathbb{R}, T_{\mathbb{R}})$. In that context, it is readily understood that a real number is a limit of a real-valued sequence precisely when the terms of the sequence get "arbitrarily close” to the number in question. This notion translates to the more general context now assumed within this definition, as the open sets in an abstract topological space are what determine “closeness” to a particular element of the underlying set. Thus, even in this setting, determining if an element is limit of a sequence can be thought of as determining if the terms of the sequence get arbitrarily close to that element!

1.3.2 Determining the Limits of Sequences

Consider the following topological space (X, T):

$$X = \{a, b, c, d\}$$

$$T = \big\{\{b\}, \{a, b\}, \{b, c, d\}, \{a, b, c, d\}, \emptyset\big\}$$

together with the following sequences in X:

$$(x_n) = (a, a, a, \dots)$$

$$(y_n) = (b, b, b, \dots)$$

$$(z_n) = (a, c, a, c, a, c, \dots)$$

In order to determine the limits of these sequences, the open sets in T must be inspected.

Beginning with (x_n), it is immediately seen that a is a limit, as every open set containing a contains all terms of (x_n). Upon further inspection, it is apparent that a is the only limit of (x_n), as the open set $\{b, c, d\}$ containing all other elements excludes a.

Examining (y_n), it is clear that a, b, c, and d are all limits, as b is an element of every open set in T. That is, all terms of (y_n) are contained in every open set in T, in which case every element of X must be a limit of (y_n). This gives way to the important observation that *limits are not necessarily unique in the general context of topology*. This stands in opposition to the behavior of limits in the more specific setting of real analysis, in which limits are in fact unique (Abbott, 2015).

Lastly, it is quickly seen that (z_n) has no limits, as the open set $\{a\}$ excludes *c* and the open set $\{b, c, d\}$ excludes a.

1.4 Continuous Functions

Now shown is the definition of a continuous function in the context of topology (Morris, 2020).

1.4.1 Definition of a Continuous Function

Definition: Let (X, T) and (Y, S) be topological spaces. Let $f: X \rightarrow Y$ be a function. To say that f is a "*continuous function*" means that if $O \in S$, then $f^{-1}[O] \in T$ (where $f^{-1}[O]$ denotes the preimage of O).

Simply put, to say that a function is continuous means that the preimage of every open set in the codomain is an open set in the domain. Though seemingly abstract at first glance, in the context of analysis, this definition is actually equivalent to the standard $\varepsilon - \delta$ definition of continuity used (Abbott, 2015)! The intuition from that setting again carries over to the general setting of topology, in that if one wishes to get "close" to the image of a point under a continuous function, one simply needs to get "close" to the original point itself.

1.4.2 An Example and a Nonexample of a Continuous Function

Consider the topological space (X, T):

$$X = \{a, b, c, d\}$$

$$T = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}, \emptyset\}$$

together with another topological space (Y, S):

$$Y = \{1,2,3,4\}$$

$$S = \{\{1\}, \{1,3\}, \{1,2,3,4\}, \emptyset\}$$

Define the function $f: X \to Y$ by:

$$f(a) = 1; f(b) = 3; f(c) = f(d) = 2$$

Examining the preimages of the open sets in S:

$$f^{-1}[\{1\}] = \{a\} \in T$$

$$f^{-1}[\{1,3\}] = \{a, b\} \in T$$

$$f^{-1}[\{1,2,3,4\}] = \{a, b, c, d\} \in T$$

$$f^{-1}[\emptyset] = \emptyset \in T$$

Thus, the preimage of every open set in S is another open set in T. Hence, f is a continuous function!

Now, consider the following function $g: X \to Y$ defined as follows:

$$g(a) = 4; g(b) = 2; g(c) = 1; g(d) = 3$$

Examining the preimages of the open sets, it can be seen that $f^{-1}[\{1\}] = \{c\} \notin T$. Thus, g is not a continuous function.

1.5 Homeomorphisms

Lastly presented in this chapter is the definition of what it means for two topological spaces to be indistinguishable (Morris, 2020).

1.5.1 Definition of a Homeomorphism

Definition: Let (X, T) and (Y, S) be topological spaces. Let $f: X \to Y$ be a function. To say that f is a "*homeomorphism*" between (X, T) and (Y, S) means that:

(a). f is a bijection.

(b). f is continuous.

(c). f^{-1} is continuous (where f^{-1} is the inverse of f).

When such a function exists, (X, T) and (Y, S) are said to be "*homeomorphic*".

Thus, a homeomorphism is a bijection between two topological spaces that is continuous in both directions. Such a map communicates that two topological spaces are structurally identical; it communicates that the two spaces only differ in the names of their points, and that their open sets are essentially the same (Morris, 2020). In this case, we consider the spaces to be indistinguishable, as in topology the naming of points is of no importance. Some may note that the concept of a homeomorphism is similar to the concept of an isomorphism in algebra, in that an isomorphism demonstrates that two *algebraic* structures are identical and differ only in the names of their points (Fraleigh, 2002).

1.5.2 Example of a Homeomorphism

Consider the same topological space (X, T) from Section 1.3.2:

$$X = \{a, b, c, d\}$$

$$T = \left\{\{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}, \emptyset\right\}$$

Define another topological space (Y, S) by

$$Y = \{1,2,3,4\}$$

$$S = \left\{\{1\}, \{2\}, \{1,2\}, \{1,2,3,4\}, \emptyset\right\}$$

X and Y are both sets of four elements, which means that there exists a bijection between them. Moreover, up to the naming of the elements, the topologies T and S appear to be identical. Thus, it is reasonable to conclude that (X, T) and (Y, S) are homeomorphic.

To prove this, consider the function $h: X \to Y$ defined as follows:

$$h(a) = 1; h(b) = 2; h(c) = 3; h(d) = 4$$

It is immediately seen that h is a bijection, and that h^{-1} is given by:

$$(h^{-1})(1) = a; (h^{-1})(2) = b; (h^{-1})(3) = c; (h^{-1})(4) = d$$

Moreover, it is not difficult to show that h is continuous:

$$h^{-1}[\{1\}] = \{a\} \in T$$

$$h^{-1}[\{2\}] = \{b\} \in T$$

$$h^{-1}[\{1,2\}] = \{a, b\} \in T$$

$$h^{-1}[\{1,2,3,4\}] = \{a, b, c, d\} \in T$$

$$h^{-1}[\emptyset] = \emptyset \in T$$

And that h^{-1} is continuous:

$$(h^{-1})^{-1}[\{a\}] = \{1\} \in T$$

$$(h^{-1})^{-1}[\{b\}] = \{2\} \in T$$

$$(h^{-1})^{-1}[\{a, b\}] = \{1,2\} \in T$$

$$(h^{-1})^{-1}[\{a, b, c, d\}] = \{1,2,3,4\} \in T$$

$$(h^{-1})^{-1}[\emptyset] = \emptyset \in T$$

Thus, h is indeed a homeomorphism between (X, T) and (Y, S). Therefore, the two spaces are homeomorphic.

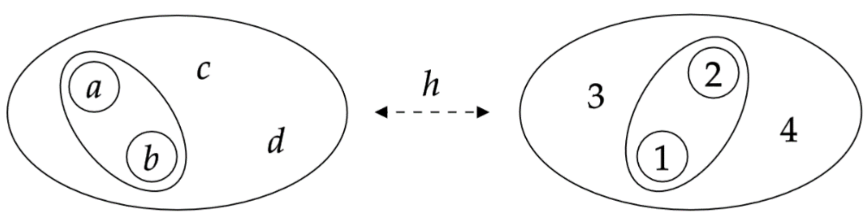

Figure 1.8: An Illustration of the Homeomorphism h

The above figure further highlights that the two spaces are identical in their topological structure.

CHAPTER 2

PARTITION SPACES

2.1 Partitions of Sets

Before proceeding, it is beneficial to define a partition of a set in the context in which it will be used for the purpose of this study (Halmos, 1987).

2.1.1 Definition of a Partition of a Set

Definition: Let X be a set. A "*partition of* X" is a disjoint collection of non-empty subsets of X, such that the union of all sets in the collection equals X.

By a disjoint collection of subsets, it is meant that for any two subsets A and B in the collection, $A \cap B = \emptyset$. Indeed, a partition of a set is what one might expect it to be: a complete decomposition of a set into disjoint pieces. Note that any nonempty set may be partitioned, whether finite or infinite.

2.1.2 Examples and Nonexamples of Partitions of Sets

Let X be the following set of symbols:

$$X = \{a, b, 1,2,3, \gamma, \delta\}$$

and let P be the following collection of subsets of X:

$$P = \left\{\{a, b\}, \{1,2,3\}, \{\gamma, \delta\}\right\}$$

It can be seen that P is in fact a disjoint collection of subsets of X, as no two sets in P have an element in common. Moreover, the union of all sets in P does in fact equal the original set X. Then, P is in fact a partition of X.

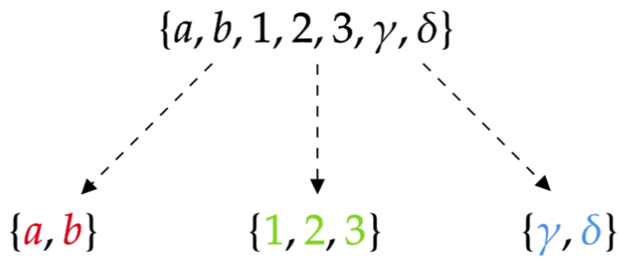

Figure 2.1: An Illustration of the Partition P

Now, let Q be the following collection of subsets of X:

$$Q = \{\{a, 1,3, \gamma\}, \{b, \delta\}\}$$

Q is indeed a disjoint collection of subsets of X. However, it is *not* true that the union of all sets in Q equals the original set X, as no set in Q contains 2 as an element. Then, Q is not a partition of X. Q can be thought of as an incomplete decomposition of X.

Consider another collection of subsets of X, given by

$$R = \{\{b, 1, \gamma\}, \{2,3\}, \{a, b, \gamma, \delta\}\}$$

Though the union of all sets in R equals X, it can be seen that R is *not* a disjoint collection of subsets of X, as the first and the third sets in R have the elements b and γ in common. As such, R is not a partition of X.

Examples of partitions of infinite sets include the even-odd decomposition of $\mathbb{N}$, given by

$$\{\{2,4,6,8, \dots\}, \{1,3,5,7, \dots\}\}$$

and the concentric circle partition of the complex plane $\mathbb{C}$, given by

$$\cup_{r \in \mathbb{R}^{\geq 0}} \{\{z \in \mathbb{C} : |z| = r\}\}$$

where $\mathbb{R}^{\geq 0}$ denotes the set of all non-negative real numbers.

2.2 Partition Spaces

The definition of the primary object of study within this work is now introduced (Seebach & Steen, 1978).

2.2.1 Definition of a Partition Space

Definition: Let X be a set. Let P be a partition of X. Let T_P be the collection of all possible unions of sets in P. T_P is the "*partition topology on X generated by P*". We call the topological space (X, T_P) a "*partition space*".

Therefore, to create a partition space using a set, one simply needs to create a partition topology on the set as described in the above definition. As a set can be partitioned in many ways, one set can give rise to many different partition topologies, and thus many different partition spaces. Note that any partition topology includes all the original sets in the partition.

It can be quickly verified that for any partition P on a set X, the partition topology T_P generated by P satisfies the first three properties needed to be a topology on X. The whole set X is an element of T_P, as the union of all sets in P equals X by the definition of a partition. The empty set, $\emptyset$, is an element of T_P, as $\emptyset$ is produced by the "empty union" of sets in P. T_P is closed under all possible unions, as any union of sets in T_P is, by construction, a union of sets in P, which is, by definition, an open set in T_P. It can also be verified that T_P is closed under all finite intersections, as one will find that any finite intersection of sets in T_P is either $\emptyset$ or a set in P.

As a final point of new terminology, the sets in a partition used to create a partition topology will be referred to as "*prime open sets*" (Phillips et al., 2022). The terminology "prime" is used in reference to the fact that the prime open sets serve as the building blocks for all other open sets in a partition topology, much like how prime numbers are viewed as the building blocks for the natural numbers. Also reminiscent of prime numbers, the terminology highlights the fact that the prime open sets are, in some sense, irreducible. For if S is a prime open set and R is an open set satisfying $R \subseteq S$, it must be that $R = S$. As can be seen, prime open sets cannot be broken down further into other open sets; they can be thought of as the most fundamental kind of open set in a partition topology. From the perspective of an element of the set used to create a partition topology, an element's prime open set is the most fundamental open set containing it, in that its prime open set is always a subset of any other open set that contains it.

2.2.2 Constructing Partition Spaces

Let X and P be defined as in Section 2.1.2:

$$X = \{a, b, 1,2,3, \gamma, \delta\}$$

$$P = \big\{\{a, b\}, \{1,2,3\}, \{\gamma, \delta\}\big\}$$

In order to create a partition topology T_P on X using P, one simply needs to collect all possible unions of sets in P:

$$T_P = \big\{\{a, b\}, \{1,2,3\}, \{\gamma, \delta\}, \{a, b, 1,2,3\}, \{a, b, \gamma, \delta\}, \{1,2,3, \gamma, \delta\}, \{a, b, 1,2,3, \gamma, \delta\}, \emptyset\big\}$$

The pair (X, T_P) is thus an example of a partition space. This process of "decomposing" X into disjoint subsets and subsequently combining them together can be visualized using the following diagram:

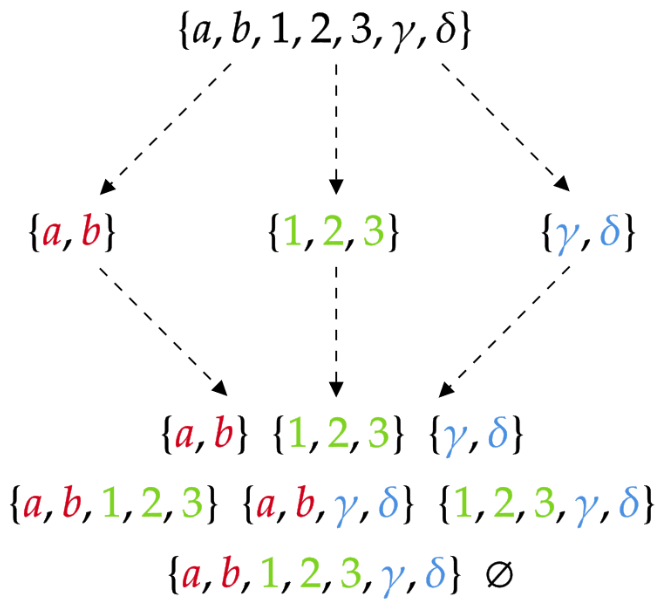

Figure 2.2: An Illustration of the Partition Topology T_P

Partition spaces involving infinite sets can be created in the same manner. For example, a partition space on $\mathbb{C}$ can be constructed using the concentric circle partition on $\mathbb{C}$ as described in Section 2.1.2. In that partition space, the prime open sets are the individual circles, and the open sets are all the possible collections of those circles.

2.3 Sequential Limits in Partition Spaces

The first result of this thesis is now introduced; it describes the behavior of limits of sequences in partition spaces. It will be demonstrated that limits of sequences in such spaces, when they exist, are bound together by prime open sets.

2.3.1 Theorem on Sequential Limits in Partition Spaces

Theorem: Let (X, T) be a partition space on a nonempty set X, let (x_n) be a sequence in X, and let L be a point in X. Let $P_L \in X$ be the prime open set containing L. Then,

(a). The intersection of all open sets containing L equals P_L.

(b). L is a limit of (x_n) if and only if P_L is the set of limits of (x_n).

Proof: a). Let I denote the index set of the collection of all open sets in T that contain L. It must be shown that $\cap_{i\in I} L_i = P_L$.

Let L_j, $j \in I$ be a particular open set in this collection. By the definition of a partition topology, L_j is a disjoint union of prime open sets in T. Let I_{L_j} denote the index set for this union, and for $k \in I_{L_j}$, let L'_k denote a prime open set in this union indexed by k. Then, $L_j = \cup_{k\in I_{L_j}} (L'_k)$.

Since this is a disjoint union of prime open sets, only one prime open set in the union contains L – this set is precisely P_L. Therefore,

$$L_j = \cup_{k\in I_{L_j}} (L'_k) = \left(\cup_{\substack{k\in I_{L_j} \\ L'_k \neq P_L}} (L'_k) \right) \cup P_L = R_j \cup P_L$$

where $R_j = \cup_{\substack{k\in I_{L_j} \\ L'_k \neq P_L}} (L'_k)$. Note that $R_j \cap P_L = \emptyset$ for all $j \in I$. Using this,

$$\cap_{i\in I} L_i = \cap_{i\in I} (R_i \cup P_L) = (\cap_{i\in I} R_i) \cup P_L$$

Since P_L is itself an open set containing L, there exists an $\alpha \in I$ such that $P_L = L_\alpha$. Then, from before, $L_\alpha = R_\alpha \cup P_L$. So $P_L = L_\alpha = R_\alpha \cup P_L$. Since $R_\alpha \cap P_L = \emptyset$, this implies that $R_\alpha = 0$. Thus,

$$\cap_{i\in I} L_i = (\cap_{i\in I} R_i) \cup P_L = \left(\left(\cap_{\substack{i\in I \\ R_i \neq R_\alpha}} R_i \right) \cap R_\alpha \right) \cup P_L = \left(\left(\cap_{\substack{i\in I \\ R_i \neq R_\alpha}} R_i \right) \cap \emptyset \right) \cup P_L$$

$$= (\emptyset) \cup P_L = P_L$$

Thus, $\cap_{i\in I} L_i = P_L$, so that the intersection of all open sets containing L equals P_L.

b). $\Rightarrow$) Suppose that L is a limit of (x_n). Every open set containing L contains all but finitely many terms of (x_n). Then, all but finitely many terms are contained within P_L.

Suppose $S \in P_L$, and let $P_S \in T$ denote the prime open set that contains S. As any two prime open sets are either disjoint or equal, this implies that $P_S = P_L$. Then, all but finitely many terms of (x_n) are contained within P_S. So, by part a), every open set containing S contains all but finitely many terms of (x_n). Therefore, S is also a limit of (x_n).

Conversely, suppose that $S \notin P_L$. Then, $P_S \cap P_L = \emptyset$. Since all but finitely many terms of (x_n) are contained within P_L, this implies that infinitely many terms of (x_n) are outside of P_S. Since P_S is an open set containing S, this implies that S is not a limit of (x_n). Thus, $S \in X$ is a limit of (x_n) if and only if $S \in P_L$. So P_L is the set of limits of (x_n).

$\Leftarrow$) Suppose that P_L is the set of limits of (x_n). It immediately follows that L is a limit of (x_n). ■

Therefore, part (b) of the theorem implies that sequences in partition spaces will either have a single prime open set as its set of limits, or absolutely no limits at all. Indeed, the limits of sequences in partition spaces are necessarily grouped together by the prime open sets. As such, to search for the limits of a particular sequence, one need only check if a *single element* from each prime open set is a limit of the sequence. If one does find that an element in a prime open set is a limit, then (b) dictates that the entire prime open set is exactly the set of limits of the sequence. Additionally, if one finds that an element in a prime open set is not a limit, then (b) guarantees that *none* of the elements in the set are limits.

Furthermore, part (a) of the theorem shows that for any element p of a set X, in a corresponding partition space (X, T), the intersection of all open sets containing p is *exactly* the prime open set containing p. Therefore, to determine if p is a limit of a sequence (x_n)

in X, one need only check if all but finitely many terms of (x_n) are contained within the *prime open set* containing p. Thus, the problem of finding the limits of sequences in partition spaces reduces to simply examining prime open sets.

2.3.2 Determining the Limits of Sequences Using Theorem 2.3.1

Consider the following set X consisting of 6 points:

$$X = \{a, b, c, d, e, f\}$$

and consider the following partition topology on X:

$$T = \{\{a, b\}, \{c, d, e\}, \{f\}, \{a, b, c, d, e\}, \{a, b, f\}, \{c, d, e, f\}, \{a, b, c, d, e, f\}, \emptyset\}$$

together with the following sequences in X:

$$(x_n) = (a, a, a, a, a, a, \dots)$$

$$(y_n) = (f, e, f, f, a, c, d, a, c, d, c, d, c, d, c, d, \dots)$$

$$(z_n) = (a, b, f, a, b, f, a, b, f, a, b, f, \dots)$$

Theorem 2.3.1 may be applied to determine whether these sequences have limits, and to quickly find those limits when they exist.

To begin, it is easily seen that a is a limit of (x_n). Then, Theorem 2.3.1 dictates that the prime open set containing a is *exactly* the set of limits of (x_n). Thus, b is also a limit of (x_n), as it is also an element of the prime open set containing a. It can then be immediately concluded that neither c, d, e, nor f are limits of (x_n), as they are not elements of the prime open set containing a.

Examining (y_n), it is clear that a is not a limit by Theorem 2.3.1, as there are infinitely many terms of value c outside of the prime open set containing a. Then, b is also not a limit of (y_n), as b is an element of that prime open set. However, c is a limit of (y_n), as all but finitely many terms of (y_n) are contained within the prime open set containing c.

Thus, d and e are also limits of (y_n), as they are both elements of that prime open set. It can then be immediately concluded that the remaining element f is not a limit of (y_n).

Considering (z_n), it can be seen that a is not a limit, as there are infinitely many terms of value f outside of the prime open set containing a. Then, b is also not a limit of (z_n). Additionally, neither c nor f are limits of (z_n), as there are infinitely many terms of value a outside of the prime open sets containing c and f, respectively. It can then be immediately concluded that d and e are also not limits of (z_n). Thus, (z_n) does not have a limit.

CHAPTER 3

MAPPINGS BETWEEN PARTITION SPACES

3.1 Continuous Functions Between Partition Spaces

The theorem now presented serves as the foundation for the rest of the results contained within this work. It completely characterizes continuous functions between partition spaces and highlights the importance of prime open sets in determining the properties of a partition space. To see why, suppose that X and Y are any two nonempty sets and that (X,T) and (Y,S) are partition spaces on them. It will be shown that, to determine if a function between X and Y is continuous, one need not consider the preimage of every single open set in S. All that needs be considered is *where the prime open sets in* T *go*.

3.1.1 The Bubble Theorem

The Bubble Theorem: Let (X,T) and (Y,S) be partition spaces. $f: X \to Y$ is a continuous function if and only if, for every prime open set $P \in T$, there exists a unique prime open set $Q \in S$ such that $f[P] \subseteq Q$.

Proof: $\Leftarrow$) Suppose that, if $P \in T$ is a prime open set, then there exists a prime open set $Q \in S$ such that $f[P] \subseteq Q$. Let O be an open set in S. Since S is a partition topology, O is equal to a union $\cup\, Q_\alpha$ of prime open sets, where $Q_\alpha \in S\ \forall \alpha$. Then,

$$f^{-1}[O] = f^{-1}[\cup\, Q_\alpha] = \cup\, f^{-1}[Q_\alpha]$$

Suppose that for all such Q_α, $f^{-1}[Q_\alpha] = \emptyset$. Then,

$$f^{-1}[O] = \cup f^{-1}[Q_\alpha] = \cup \emptyset = \emptyset \in T$$

in which case $f^{-1}[O]$ is open in T.

Now, suppose that there exists a Q_α such that $f^{-1}[Q_\alpha] \neq \emptyset$. Let $x \in f^{-1}[Q_\alpha]$. Since T is a partition topology, there exists a prime open set $P_x \in T$ such that $x \in P_x$. By hypothesis, there exists a prime open set $Q_x \in S$ such that $f[P_x] \subseteq Q_x$. Since $f(x) \in Q_\alpha$ and $f(x) \in Q_x$, and since Q_α and Q_x are both prime open sets in S, it must be the case that $Q_\alpha = Q_x$. Then, $f[P_x] \subseteq Q_\alpha$. Thus, $P_x \subseteq f^{-1}[Q_\alpha]$.

Then, for any $x \in f^{-1}[Q_\alpha]$, the prime open set P_x containing x is such that $P_x \subseteq f^{-1}[Q_\alpha]$. This in turn implies that, for any $y \notin f^{-1}[Q_\alpha]$, the prime open set P_y containing y is such that $P_y \cap f^{-1}[Q_\alpha] = \emptyset$ - if this intersection were non-empty, then the entire set P_y containing y would be a subset of $f^{-1}[Q_\alpha]$. Together, these imply that $f^{-1}[Q_\alpha]$ is *exactly* a union $\cup P$ of prime open sets $P \in T$. Then,

$$f^{-1}[O] = \cup f^{-1}[Q_\alpha] = \cup (\cup P)$$

which is a union of prime open sets in T. Thus, $f^{-1}[O]$ is an open set in T. Thus, f is continuous.

$\Rightarrow$) Suppose that f is continuous. Let $Q \in S$ be a prime open set. Since f is continuous, $f^{-1}[Q]$ is an open set in T. Since T is a partition topology, $f^{-1}[Q] = \cup P$, where each P is a prime open set in T. Then, for each P, $f[P] \subseteq Q$.

If we repeat this argument for every prime open set $Q \in S$, we include all possible prime open sets in T, as the union of all such Q equals the whole codomain S. Then, for every prime open set $P \in T$, there exists a prime open set $Q \in S$ such that $f[P] \subseteq Q$.

For uniqueness, suppose that there exist prime open sets $P \in T$ and $Q, Q' \in S$ satisfying $f[P] \subseteq Q$ and $f[P] \subseteq Q'$. Let $x \in P$. Since $f[P] \subseteq Q$, $f(x) \in Q$. Since $f[P] \subseteq Q'$, $f(x) \in Q'$. Since Q and Q' are prime open sets, this implies that $Q = Q'$. Thus, Q is unique in this manner. ■

Therefore, there is a special relationship between continuous functions and partition spaces: if one wishes to define a continuous function between such spaces, all one must do is map the points in each prime open set of the domain into exactly one prime open set of the codomain. It is as if the individual points in the prime open sets do not matter! Veritably, all that matters is that the prime open sets in the domain are kept intact. Thus, continuous functions treat prime open sets in the domain like bubbles, sending them over or shrinking them down into other prime open sets in the codomain. Intuitively, this seems like a "continuous" action: such functions bend and stretch prime open sets without ever breaking them.

3.1.2 Constructing a Continuous Function Using the Bubble Theorem

The Bubble Theorem may be applied to directly construct a continuous function between partition spaces. Let X again be a set of six points:

$$X = \{a, b, c, d, e, f\}$$

with the same partition topology T as in Section 2.3.1:

$$T = \{\{a, b\}, \{c, d, e\}, \{f\}, \{a, b, c, d, e\}, \{a, b, f\}, \{c, d, e, f\}, \{a, b, c, d, e, f\}, \emptyset\}$$

Let Y be the following set of eight points:

$$Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

together with the partition topology S:

$$S = \{\{1,3,6,7,8\}, \{2,5\}, \{4\}, \{1,2,3,5,6,7,8\}, \{1,3,4,6,7,8\}, \{2,4,5\}, \{1,2,3,4,5,6,7,8\}, \emptyset\}$$

In order to define a continuous function g between X and Y, one need only map the elements of each prime open set in T into exactly one prime open set in S. One such choice of mapping for the prime open sets in T is:

$$\{a, b\} \xrightarrow{g} \{4\}$$

$$\{c, d, e\} \xrightarrow{g} \{2,5\}$$

$$\{f\} \xrightarrow{g} \{1,3,6,7,8\}$$

So, for example, for the function $g: X \to Y$ defined as follows:

$$g(a) = g(b) = 4;\ g(c) = 2;\ g(d) = 5;\ g(e) = 2;\ g(f) = 8$$

the Bubble Theorem guarantees that g is continuous! This mapping can be visualized through the following figure:

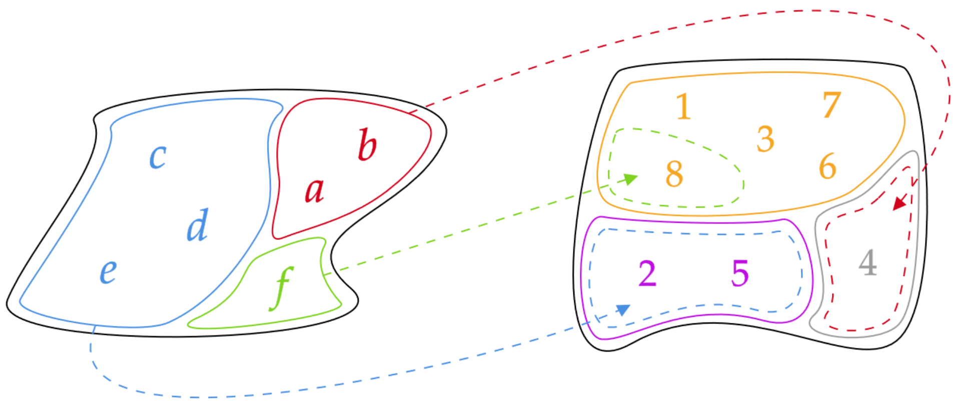

Figure 3.1: An Illustration of the Continuous Function g

Each partition space is represented in the diagram by superimposing their prime open sets on top of the underlying set. Indeed, the prime open sets look like bubbles in their partition spaces. g bends and stretches each bubble in the domain into another bubble in the codomain.

3.1.3 Showing That a Function is Not Continuous Using the Bubble Theorem

In addition to being a useful tool in showing that a function between partition spaces is continuous, the Bubble Theorem makes it easy to show that a function between such spaces is not continuous. To show that a function between such spaces is not continuous, all that one must demonstrate is that the points of a particular prime open set in the domain map to points in more than one prime open set in the range. For example, consider the same partition spaces (X, T) and (Y, S) as in Section 3.1.2:

$$X = \{a, b, c, d, e, f\}$$

$$T = \{\{a, b\}, \{c, d, e\}, \{f\}, \{a, b, c, d, e\}, \{a, b, f\}, \{c, d, e, f\}, \{a, b, c, d, e, f\}, \emptyset\}$$

$$Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$S = \{\{1,3,6,7,8\}, \{2,5\}, \{4\}, \{1,2,3,5,6,7,8\}, \{1,3,4,6,7,8\}, \{2,4,5\}, \{1,2,3,4,5,6,7,8\}, \emptyset\}$$

Now, define the function $q: X \to Y$ as follows:

$$q(a) = 2; q(b) = 5; q(c) = q(d) = 4; q(e) = 7; q(f) = 1$$

The elements of $\{a, b\} \in T$ map to elements of $\{2,5\} \in S$. However, the elements of $\{c, d, e\} \in T$ map to elements of $\{4\} \in S$ *and* to elements of $\{1,3,6,7,8\} \in S$.

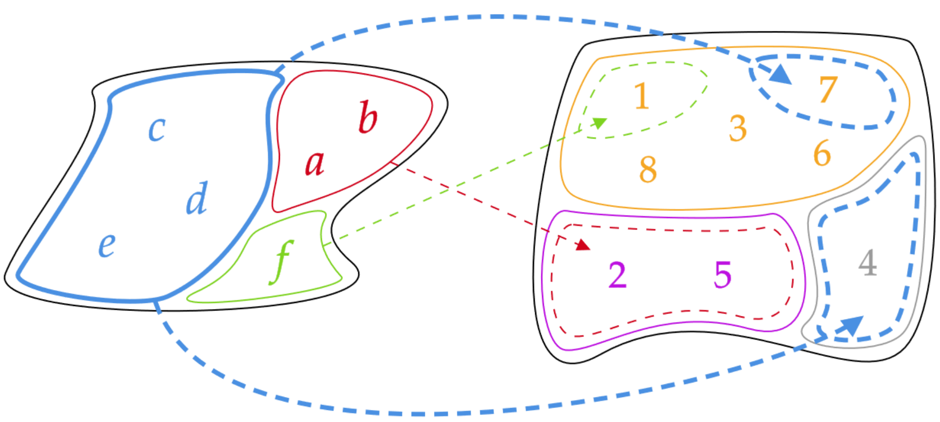

Figure 3.2: An Illustration of the Non-Continuous Function q

Thus, a prime open set in the domain maps to two different prime open sets in the range. By the Bubble Theorem, it can then be immediately concluded that q is not continuous! q can be thought of as "popping" the bubble formed by $\{c, d, e\}$ by mapping its elements to elements of two different prime open sets in S.

3.2 Continuous Functions and Sequential Limits in Partition Spaces

Now presented is a result detailing how continuous functions between partition spaces interact with the limits of sequences in their domains. It will be shown that the limits of the image of a sequence under a continuous function are entirely determined by the limits of the original sequence itself. This draws similarity to the relationship between continuous functions and limits of sequences in the context of real analysis (i.e., the sequential criterion for continuity) (Abbott, 2015).

3.2.1 Theorem on Continuous Functions and Sequential Limits in Partition Spaces

Theorem: Let (X, T) and (Y, S) be partition spaces. Let $f: X \to Y$ be a continuous function. Let (x_n) be a sequence in X. Suppose that (x_n) has a limit in X, so that it has the prime open set $P \in T$ as its set of limits. Then, the unique prime open set $Q \in S$ satisfying $f[P] \subseteq Q$ is the set of limits of the sequence $(f(x_k))$.

Proof: Since P is the set of limits of (x_n), all but finitely many terms of (x_n) are contained in P. This implies that all but finitely many terms of the sequence $(f(x_k))$ are contained in Q.

Let $L \in Q$. As Q is a prime open set, part (a) of Theorem 2.3.1 implies that every open set in S containing L contains all but finitely many terms of $(f(x_k))$. So L is a limit of $(f(x_k))$. Part (b) of Theorem 2.3.1 then implies that Q is exactly the set of limits of $(f(x_k))$. ■

In some sense, continuous functions between partition spaces preserve the limits of sequences in their domains. For if the prime open set of limits of a sequence in a partition space is known, the prime open set of limits for the image of the sequence under a continuous function is known. Note that the theorem makes no comment on the *number* of limits of this image sequence. It will be demonstrated that such a continuous function can map a sequence in its domain to a new sequence in its codomain that possesses either an equal or a different number of limits. It is also worth noting that the theorem *does not state* that such a continuous function will map a sequence in its domain having no limits to another sequence in its codomain having no limits. It will also be demonstrated that such a continuous function can map a sequence in its domain that has no limits to a sequence in its codomain that does have a limit.

3.2.2 Determining the Limits of the Continuous Image of a Sequence Using Theorem 3.2.1

Again, consider the partition spaces (X, T) and (Y, S) as defined in Section 3.1.2:

$$X = \{a, b, c, d, e, f\}$$

$$T = \{\{a, b\}, \{c, d, e\}, \{f\}, \{a, b, c, d, e\}, \{a, b, f\}, \{c, d, e, f\}, \{a, b, c, d, e, f\}, \emptyset\}$$

$$Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$S = \{\{1,3,6,7,8\}, \{2,5\}, \{4\}, \{1,2,3,5,6,7,8\}, \{1,3,4,6,7,8\}, \{2,4,5\}, \{1,2,3,4,5,6,7,8\}, \emptyset\}$$

together with the continuous function $g: X \to Y$ from Section 3.1.2:

$$g(a) = g(b) = 4;\ g(c) = 2;\ g(d) = 5;\ g(e) = 2;\ g(f) = 8$$

and the following sequences in X, as defined in Section 2.3.2:

$$(x_n) = (a, a, a, a, a, a, \ldots)$$

$$(y_n) = (f, e, f, f, a, c, d, a, c, d, c, d, c, d, c, d, \ldots)$$

$$(z_n) = (a, b, f, a, b, f, a, b, f, a, b, f, \ldots)$$

It was determined in Section 2.3.2 that (x_n) had the prime open set of limits $\{a, b\}$, that (y_n) had the prime open set of limits $\{c, d, e\}$, and that (z_n) had no limits.

The images of these sequences under g are as follows:

$$\big(g(x_n)\big) = (4,4,4,4,4,4, \dots)$$

$$\big(g(y_n)\big) = (8,2,8,8,4,2,5,4,2,5,2,5,2,5,2,, \dots)$$

$$\big(g(z_n)\big) = (4,4,8,4,4,8,4,4,8,4,4,8, \dots)$$

Using Theorem 2.3.1, it can be seen that $\big(g(x_n)\big)$ has the prime open set of limits $\{4\}$, which is precisely the prime open set in Y that g maps $\{a, b\}$ into.

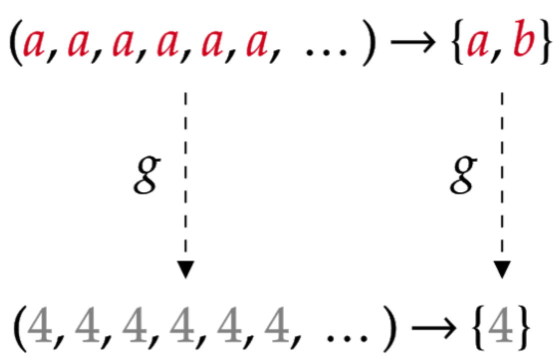

Figure 3.3: The Image of the Sequence (x_n) Under the Function g, Together With the Image of Its Prime Open Set of Limits

In the same way, it can be seen that $\big(g(y_n)\big)$ has the prime open set of limits $\{2,5\}$, which is the prime open set that g maps $\{c, d, e\}$ into.

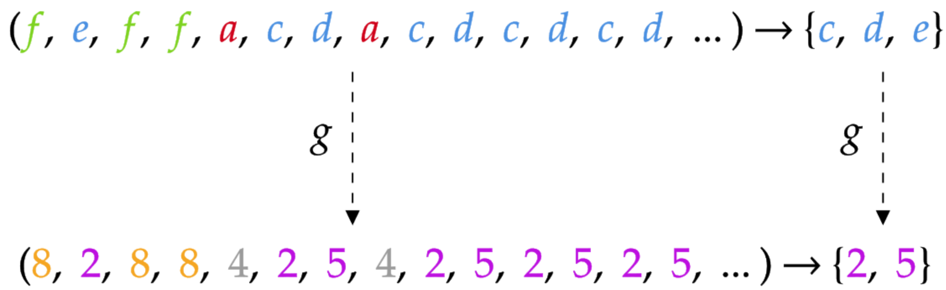

Figure 3.4: The Image of the Sequence (y_n) Under the Function g, Together With the Image of Its Prime Open Set of Limits

Notice that the number of limits of $(g(y_n))$ is *less than* the number of limits of (y_n). Using Theorem 2.3.1, it is also quickly verified that $(g(z_n))$ has no limits in Y.

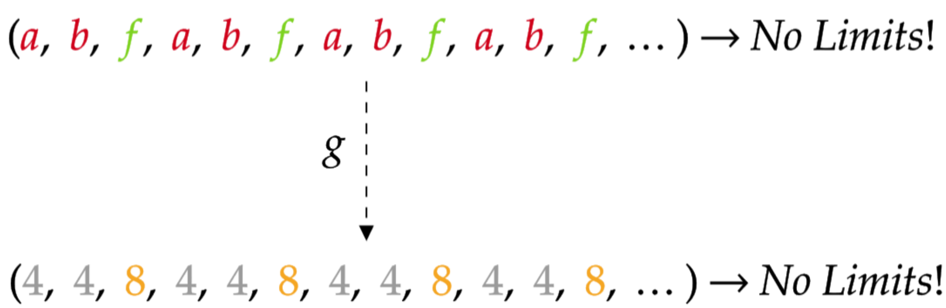

Figure 3.5: The Image of the Sequence (z_n) Under the Function g

However, as previously stated, it could very well be that a continuous function between (X,T) and (Y,S) maps (z_n) to a sequence in Y having a limit. For example, consider the function $g': X \to Y$ defined as follows:

$$g'(a) = g'(b) = 3;\ g'(c) = 2;\ g'(d) = 5;\ g'(e) = 2;\ g'(f) = 8$$

g’ can be visualized using the following diagram:

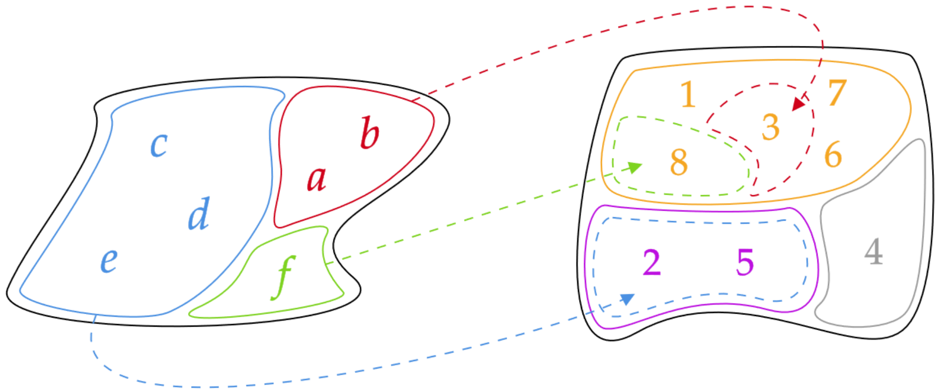

Figure 3.6: An Illustration of the Function g'

The image of (z_n) under g' is given by

$$\left(g'(z_n)\right) = (3,3,8,3,3,8,3,3,8,3,3,8,\ldots)$$

which, by Theorem 2.3.1, has the prime open set of limits $\{1,3,6,7,8\}$ in Y.

3.3 Homeomorphisms Between Partition Spaces

The Bubble Theorem demonstrates an interesting connection between continuous functions and prime open sets. Since the definition of a homeomorphism between topological spaces is based upon continuous functions, one might predict that the Bubble Theorem can be used to describe homeomorphisms between partition spaces from the perspective of prime open sets. As will be demonstrated, the Bubble Theorem can indeed be used to derive an alternative set of necessary and sufficient conditions for a function between partition spaces to be a homeomorphism, which are based upon the images of prime open sets.

3.3.1 Theorem on Homeomorphisms Between Partition Spaces

Theorem: Let (X,T) and (Y,S) be partition spaces. A function $f: X \to Y$ is a homeomorphism if and only if:

(a). f is a bijection, and

(b). for every prime open set $P \in T$, there exists a prime open set $Q \in S$ such that $f[P] = Q$.

Proof: From the definition of a homeomorphism, f is a homeomorphism if and only if the following hold:

(a). f is a bijection.

(i). f is continuous

(ii). f^{-1} is continuous

Using the Bubble Theorem, these statements are equivalent to the following:

(a). f is a bijection.

(I). For all prime open sets $P \in T$, there exists a prime open set $Q \in S$ such that $f[P] \subseteq Q$.

(II). For all prime open sets $Q \in S$, there exists a prime open set $R \in T$ such that $(f^{-1})[Q] \subseteq R$.

Thus, to prove the corollary, one need only show that Statements (a), (I), and (II) together are equivalent to (a) and (b) together.

⇒) First, suppose that (a), (I), and (II) hold. Let $P' \in T$ be a prime open set. Then, there exists a prime open set $Q' \in S$ such that $f[P'] \subseteq Q'$. Then, $P' \subseteq (f^{-1})[Q']$. As Q' is a prime open set, there also exists a prime open set $R' \in T$ such that $(f^{-1})[Q'] \subseteq R'$. Then, $P' \subseteq (f^{-1})[Q'] \subseteq R'$, in which case $P' \subseteq R'$. Since R' is a prime open set in T, this implies that $P' = R'$. Then, $P' \subseteq (f^{-1})[Q'] \subseteq P'$, which implies that $(f^{-1})[Q'] = P'$. Thus, $f[P'] = Q'$. Thus, for every prime open set $P \in T$, there exists a prime open set $Q \in S$ such that $f[P] = Q$. So (b) holds. Hence, Statements (a), (I), and (II) imply Statements (a) and (b).

⇐) Now, suppose that (a) and (b) hold. As it is assumed that for all prime open sets $P \in T$, there exists a prime open set $Q \in S$ such that $f[P] = Q$, it immediately follows that for all prime open sets $P \in T$, there exists a prime open set $Q \in S$ such that $f[P] \subseteq Q$. Thus, (I) holds.

This assumption also implies that for all prime open sets $P \in T$, there exists a prime open set $Q \in S$ such that $P = (f^{-1})[Q]$. As the union $\cup P$ of all such P equals X,

$$Y = f[X] = f[\cup P] = f[\cup (f^{-1})[Q]] = \cup f[(f^{-1})[Q]] = \cup Q$$

In this way, all such prime open sets $Q \in S$ are included. So, for all prime open sets $Q \in S$, there exists a prime open set $P \in T$ such that $(f^{-1})[Q] = P$. It immediately follows that $(f^{-1})[Q] \subseteq P$. Thus, (II) holds. Hence, Statements (a) and (b) imply Statements (a), (I), and (II). ■

The theorem states that, for partition spaces (X, T) and (Y, S), a bijection f between X and Y is a homeomorphism exactly when f maps each prime open set in T onto exactly one prime open set in S. In that case, f simply "renames" the elements of each prime open set in T using the elements of exactly one prime open set in S. Then, f demonstrates not only that X and Y are essentially the same set, but also that the partitions on X and Y used to create the topologies T and S are essentially the same. Since the partitions on the sets define the corresponding partition spaces (X, T) and (Y, S), in this way, f communicates that the spaces *have the exact same topological structure*. That is, f conveys that the two spaces are essentially the same, just with different names for their points.

This is exactly the notion captured by a homeomorphism, and it is reflected beautifully in the context of partition spaces. Since prime open sets entirely determine the structure of a partition space, if the prime open sets within two partition spaces are the same up to renaming of the elements, then the spaces in their entirety must be topologically the same!

3.3.2 Demonstrating That Partition Spaces Are Homeomorphic Using Theorem 3.3.1

Theorem 3.3.1 simplifies the task of demonstrating that two partition spaces are indeed homeomorphic. For example, let X be the following set of eight numbers:

$$X = \{1,3,5,7,9,11,13,15\}$$

together with the following partition topology T:

$$T = \{\{1,5\}, \{3,13\}, \{7,11,15\}, \{9\}, \{1,3,5,13\}, \{1,5,7,11,15\}, \{1,5,9\},$$

$$\{3,7,11,13,15\}, \{3,9,13\}, \{7,9,11,15\}, \{1,3,5,7,9,11,13,15\}, \emptyset\}$$

Let Y be another set of eight numbers:

$$Y = \{2,4,6,8,10,12,14,16\}$$

together with the following partition topology S:

$$S = \{\{2,8,16\}, \{4,6\}, \{10\}, \{12,14\},\{2,4,6,8,16\}, \{2,8,10,16\}, \{2,8,12,14,16\},$$

$$\{4,6,10\}, \{4,6,12,14\}, \{10,12,14\}, \{2,4,6,8,10,12,14,16\}, \emptyset \}$$

At first glance, the spaces (X, T) and (Y, S) may appear to be unrelated. Their elements are named differently, and their topologies do not immediately look the same.

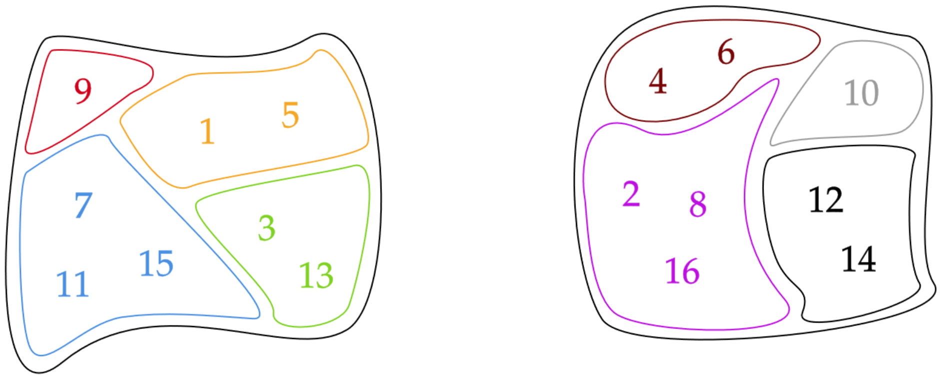

Figure 3.7: Two Partition Spaces That May Initially Seem Topologically Distinct

However, notice that both X and Y are sets of eight elements, implying that there exists a bijection between X and Y. Furthermore, notice that both T and S contain one prime open set of one element, two prime open sets of two elements, and one prime open set of three elements. These observations suggest that the spaces are in fact homeomorphic.

To prove this, one need only come up with a specific homeomorphism between the two spaces; using Theorem 3.3.1, one need only create a bijection $H: X \to Y$ that sends the elements of each prime open set in T to the elements of exactly one prime open set in S. One such choice of mapping for the prime open sets is

$$\{1,5\} \xrightarrow{H} \{4,6\}$$

$$\{3,13\} \xrightarrow{H} \{12,14\}$$

$$\{7,11,15\} \xrightarrow{H} \{2,8,16\}$$

$$\{9\} \xrightarrow{H} \{10\}$$

Thus, if H is defined as follows:

$$H(1) = 4;\ H(5) = 6;\ H(3) = 12;\ H(13) = 14;\ H(7) = 2;$$

$$H(11) = 8;\ H(15) = 16;\ H(9) = 10$$

H is guaranteed to be homeomorphism by Theorem 3.3.1.

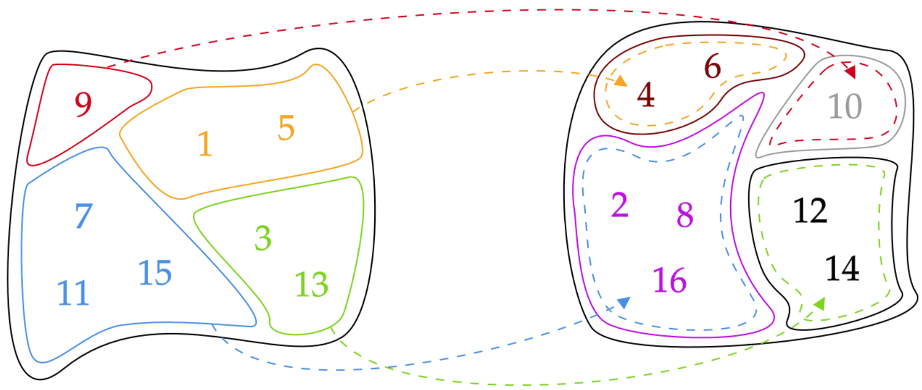

Figure 3.8: An Illustration of the Homeomorphism H

With the visual aid of the diagram, the structural similarities of the two spaces are now clearly reflected. Though the bubbles around the points may be drawn differently, they describe the same partition space.

3.3.3 Demonstrating That Partition Spaces Are Not Homeomorphic Using Theorem 3.3.1

In addition to making the construction of homeomorphisms between partition spaces simpler, Theorem 3.3.1 also reduces the work needed to show that two partition spaces are *not* homeomorphic. Consider the same partition space (X, T) from Section 3.3.2:

$$X = \{1,3,5,7,9,11,13,15\}$$

$$T = \{\{1,5\},\{3,13\},\{7,11,15\},\{9\},\{1,3,5,13\},\{1,5,7,11,15\},\{1,5,9\},$$

$$\{3,7,11,13,15\},\{3,9,13\},\{7,9,11,15\},\{1,3,5,7,9,11,13,15\},\emptyset\}$$

Let Y be the same set of eight numbers from Section 3.3.2:

$$Y = \{2,4,6,8,10,12,14,16\}$$

but define a new partition topology R on Y:

$$R = \{\{2,14\},\{4,8,12,16\},\{6,10\},\{2,4,8,12,14,16\},\{2,6,10,14\},\{4,6,8,10,12,16\}$$

$$\{2,4,6,8,10,12,14,16\},\emptyset\}$$

Upon inspection, it can be seen that R has a prime open set containing four elements. In order to map each prime open set in T to exactly one prime open set in R using a bijection, T would then need to have a prime open set containing exactly four elements. However, there are no prime open sets in T that contain exactly four elements. Hence, by Theorem 3.3.1, it is impossible for *any* bijection between X and Y to be a homeomorphism between (X, T) and (Y, S). Thus, it can be concluded that (X, T) and (Y, S) are not homeomorphic! This conclusion supports the intuition of homeomorphisms describing structural similarity: the two spaces are fundamentally different, having been constructed from partitions of different structure.

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BIOGRAPHICAL INFORMATION

Gabriel Cantanelli began his collegiate career at the University of Texas at Arlington (UTA) in the fall of 2018, where he has worked to complete an Honors Bachelor of Science in Mathematics and a Second Major in Physics. He is an avid student of the sciences and has pursued both coursework and research in a variety of fields, including chemistry, physics, and mathematics. He enjoys learning about topics lying at the intersection of the theoretical and the physical, and loves finding new ways to visually understand ideas. He also hopes to attend graduate school in the future, where he intends to pursue a doctoral degree in applied mathematics.

Gabriel has also been a part of the Honors College since his very first semester at UTA, and greatly enjoys being a part of the honors student community. He has served as an Honors Advocate for two and a half years, and as a Lead Advocate for a year and a half. He has also helped to initiate and develop an Advocate student mentorship program, which enabled him to co-present on creating peer-mentoring programs at the 2022 National Collegiate Honors Council Conference in Dallas, Texas.

In addition to his academic interests, some of Gabriel's favorite pastimes include weightlifting, cooking, and reading. He also enjoys fashion and shopping for antique paintings.