Spring 2024

Higher-Derivative Quantum Field Theory and Its Implications for Hawking Radiation and Nonlocality

Gordon Kanan
University of Texas at Arlington

HIGHER-DERIVATIVE QUANTUM FIELD THEORY AND ITS IMPLICATIONS FOR HAWKING RADIATION AND NONLOCALITY

by

GORDON KANAN

Presented to the Faculty of the Graduate School of
The University of Texas at Arlington in Partial Fulfillment
of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON
May 2024

ACKNOWLEDGEMENTS

First I would like to thank my doctoral adviser, Dr. Zdzislaw Musielak, for the guidance through a conceptually difficult topic and his recommendations of certain valuable resources. His help on the organization of the thesis was essential to constructing a coherent result. I also want to thank Dr. John Fry for the suggestions and advice regarding aspects of my thesis. Some of his questions aided me in clarifying (at least in my own mind) certain points in my thesis. I especially want to thank both Dr. Musielak and Dr. Fry for their warm friendship which I have greatly appreciated.

I would also like to express my gratitude to Dr. Alex Weiss, Dr. Andrew White and Dr. Muhammad Huda for being members of the dissertation committee and for their questions and time in participating in the oral defense of my thesis.

Finally, I want to thank my wife for her patience and tolerance and for our 42 years of happy, married life. Without her I would never have been able to fulfill one of my long and passionate goals, the study of physics.

May 28,2024

ABSTRACT

HIGHER-DERIVATIVE QUANTUM FIELD THEORY AND ITS IMPLICATIONS FOR HAWKING RADIATION AND NONLOCALITY

Gordon Kanan, Ph.D.

The University of Texas at Arlington, 2024

Supervising Professor: Zdzislaw Musielak

One of the fundamental equations of quantum field theory is the Klein-Gordon equation which can be constructed using irreducible representations of the Poincaré group and describes the dynamics of spin-0 matter. The higher derivative Klein-Gordon equations are also constructed using irreducible representations of the Poincaré group and are, thus, invariant under operations of this group. These higher derivative Klein-Gordon equations can be placed into two series depending on the power of the derivative, one for odd powers of the derivative and one for even powers, whose solutions yield timelike and spacelike fields. Applying these higher derivative equations to a Schwarzschild black hole allows investigation of massless and massive particle emissions in addition to the known Hawking radiation, as well as implying a flux of tachyonic quantum fields from the black hole. The spacelike fields deduced from the higher derivative Klein-Gordon equation offer a possible explanation of nonlocality, as in the case of entangled particles.

TABLE OF CONTENTS

CHAPTER 1

Introduction

1.1 Brief Historical Background

During the late 1600's and early 1700's Newton considered light to have a particle nature, even as in approximately the same period Robert Hooke, Christiaan Huygens and Augustin-Jean Fresnel were developing a wave theory of light. The Thomas Young double-slit experiment in 1801 showed wave interference of light adding weight to the wave view of light. At the end of the 18th century light was considered strictly as a wave as described by Maxwell's equations and Maxwell showed that visible light, infrared light, and ultraviolet light were all electromagnetic waves of differing frequencies. But in 1900 Planck proposed that light was emitted in black-body radiation as discrete quanta of energy and somewhat later Einstein, also, suggested that light is emitted and absorbed as discrete quanta in his work on the photoelectric effect. Niels Bohr published his famous paper on the hydrogen atom in 1913 and in that paper he described the laws for the hydrogen atom's spectral lines. Throughout the 1910's and 1920's a host of prominent physicists such as Born, Planck, Heisenberg, Dirac, Pauli, Bohr and Einstein to name a few were developing the early quantum mechanics. In 1924 Louis de Broglie published his doctoral thesis, for which he won the Nobel prize, proposing that matter can exhibit wave properties. Using de Broglie's work Erwin Schrödinger produced his equation for the time evolution of a quantum mechanical wave in 1926. Max Born gave the interpretation of the square of the absolute value of the wave as being the probability density amplitude of the property under consideration.

The first formulation of quantum mechanics (QM) was non-relativistic quantum mechanics (NRQM) as represented by the Schrödinger equation (Schrödinger, 1928)

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi + V\phi, \tag{1.1}$$

where ϕ is the wave function solution and V is the potential. This equation was obtained by first quantization of classical equations. Any physical theory must be consistent with special relativity and soon after the development of NRQM physicists began working on a theory to incorporate special relativity into QM leading to relativistic quantum mechanics (RQM). This effort quickly led to quantum field theory (QFT) using second quantization. It became immediately apparent that spin was fundamentally important in QFT and different equations were needed to describe spin-0 particles (Klein-Gordon equation), spin-1/2 particles (Dirac equation) and spin-1 particles (Proca equation and Maxwell's equations) (Greiner,1990; Ryder, 1996).

Eugene Wigner made significant contributions to the development of quantum mechanics through the use of group theory. In particular, he explained that SU(4), the special unitary group in four dimensions, could be applied to nuclear forces. But even more importantly, working on the Lorentz group, he determined a set of irreducible unitary representations which led to his famous 1939 paper on representations of the inhomogeneous Lorentz group (Poincaré group) which allowed for the identification of elementary particles with labels for mass and spin (Wigner, 1939).

1.2 Basic Quantum Field Theory Equations

The Klein-Gordon (K-G) equation for spin-0 particles in QFT is a differential equation which is 2nd order in time and 2nd order in the spatial coordinates and whose solutions are scalar wave functions. The form of the K-G equation is

$$(\partial^{\mu}\partial_{\mu} + m^2)\psi(x) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(x) = 0, \qquad (1.2)$$

where $x = (x^0, x^1, x^2, x^3)$ and $\psi(x)$ is the wave function solution. Later we use ω_0^2 in place of m^2 which will be explained in section 3.2. Since the K-G equation is consistent with special relativity the metric used is the Minkowski metric in flat spacetime (Klein, 1926; Gordon, 1926)

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \qquad (c = 1). \qquad (1.3)$$

In 1927 Dirac began searching for an equation combining special relativity with quantum mechanics resulting in the Dirac equation for fermions. At the same time many physicists began applying quantum mechanics to fields instead of individual particles which led to quantum field theory. The Dirac equation for spin-1/2 particles is

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0, \qquad (1.4)$$

where the γ^{μ} 's are the Dirac matrices or the γ-matrices, which are 4x4 matrices. The solutions to the Dirac equation are 4-component spinors, ψ, involving energy, E, momentum, $\mathbf{p}$, and mass, m (Dirac, 1928).

The Proca equations for spin-1 particles are

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}; \quad \partial_{\mu}F^{\mu\nu} + m^2 A^{\nu} = 0; \quad \partial_{\nu}A^{\nu} = 0, \qquad (1.5)$$

where $F^{\mu\nu}$ = electromagnetic field tensor and A^{ν} = 4-vector potential.

When $m = 0$, as in the case of photons, the Proca equations reduce to Maxwell's equations. The solutions to the Proca and Maxwell's equations are vector wave functions (Proca, 1936).

1.3 Outline of Dissertation

This thesis will deal only with spin-0 particles using a higher-derivative K-G equation (Musielak et al., 2015). For reference in the rest of this section the K-G equation is $(\partial^\mu \partial_\mu + k^\mu k_\mu)\psi(x) = 0$. In essence, the higher-derivative K-G equation results in two sequences of increasing powers of the K-G equation by repeated application of the eigenvalue equation $i\partial^\mu \psi = k^\mu \psi$, where $\partial^\mu = \partial/\partial_\mu = (\partial_t, -\nabla)$ and $k^\mu = (\omega, k^i)$. This will be explained in greater detail in Chapter 3.

We will be concerned only with real solutions to these two series which would represent physically meaningful solutions. The other solutions to a given higher order K-G equation are either complex solutions or purely imaginary solutions. One series of these higher-derivative K-G equations results only in the original K-G equation with its well-known solutions. However, the other series results in the original K-G equation and, additionally, a K-G equation with a negative value for the constant term. The solution to the second equation implies the existence of tachyonic fields for which a quantum field theory for free spin-0 fields is explored in Chapter 4. In Chapter 5 the "tachyonic" equation is studied for possible addition to Hawking radiation from Schwarzschild black holes. Finally, in Chapter 6 the Einstein/Podolski/Rosen (EPR) paper, Bell's theorem and the 2022 Nobel prize in physics awarded to Alain Aspect, John Clauser and Anton Zeilinger for their work on nonlocality are discussed with consideration of the possible relation between nonlocality and tachyonic fields in quantum mechanics.

CHAPTER 2

Quantum Field Theory for Scalar Fields

2.1 Quantum Field Theory

The Klein-Gordon (K-G) equation, Dirac equations and Proca/Maxwell equations are the basis for Quantum Field Theory (QFT). The principle of least action, Lagrange density and Hamiltonian density are also fundamental to QFT. Since this paper deals only with spin-0 fields, only the K-G equation will be considered in what follows (Ryder, 1996; Greiner and Reinhardt, 1996).

2.2 Real Klein-Gordon Field

As described above the K-G equation is

$$(\partial^\mu \partial_\mu + m^2)\psi(x) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \psi(x) = 0. \tag{2.1}$$

Second quantization is performed by keeping the same commutators from Non-Relativistic Quantum Mechanics (NRQM) and using the relativistic Lagrangian (or relativistic Hamiltonian). This, then, gives a relativistic quantum mechanical equation where $\psi(x)$ is a state or wave function. There are two problems with this Relativistic Quantum Mechanics (RQM) conception. One is that the probability density is not positive definite. The other is the appearance of negative energy states. These two problems are solved by considering the K-G equation as a field equation in which $\phi(x)$, the Fourier transform of $\psi(x)$, now becomes a field operator (Greiner and Reinhardt, 1996).

2.2.1 Principle of Least Action

To go from RQM to QFT the Lagrange density from classical field theory is used and canonical commutation relations are applied. The Hamiltonian density can be obtained from the Lagrange density using the Legendre transformation. The canonical commutation relation becomes

$$[\phi(x), \pi(x')] = i\hbar\delta(x - x'), \tag{2.2}$$

where $\phi(x)$ is now a field operator and $\pi(x)$ is the conjugate momentum operator.

2.2.2 Solutions to the K-G equation

The solutions to the real K-G equation (since we are considering only electrically neutral particles) can be written

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}], \tag{2.3}$$

where $k = k_\mu = (\omega_k, \mathbf{k})$, $kx = k_\mu x^\mu$ and $a(k)$ is an annihilation operator and $a^\dagger(k)$ is a creation operator. These operators obey the following commutation relations (Ryder, 1996)

$$[a(k), a^\dagger(k')] = \delta(\mathbf{k} - \mathbf{k}') \tag{2.4}$$

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0. \tag{2.5}$$

2.2.3 Lagrangian density, Hamiltonian density, Number operator

The Lagrangian density for a real, relativistic, scalar field is

$$\mathcal{L} = \frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2]. \tag{2.6}$$

The Hamiltonian density is obtained from the Lagrangian using the Legendre transformation

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}[\partial_\mu\phi\partial^\mu\phi + m^2\phi^2] \tag{2.7}$$

and the Hamiltonian is then found by integrating the Hamiltonian density

$$H = \int \mathcal{H} d^3x = \frac{1}{2} \int \left[(\partial_0 \phi)^2 + \nabla \phi \cdot \nabla \phi + m^2 \phi^2 \right] d^3x. \tag{2.8}$$

After substitution for the ϕ's and considerable algebra the Hamiltonian is shown to be

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{\omega_k}{2} [a^\dagger(k)a(k) + a(k)a^\dagger(k)]. \tag{2.9}$$

The number operator, $N(k)$, is defined to be $N(k) = a^\dagger(k)a(k)$ and it is easy to show that $[N(k), N(k')] = 0$. Operating on an eigenstate the number operator will give the number of particles in that state, i.e. $N(k)|n(k)\rangle = n(k)|n(k)\rangle$.

The Hamiltonian can now be written in terms of the number operator

$$H = \int d^3k \; \omega_k \left[N(k) + \frac{1}{2} \right]. \tag{2.10}$$

2.2.4 Annihilation and Creation Operators

As mentioned above $a(k)$ is the annihilation operator and $a^\dagger(k)$ is the creation operator. They operate on states in the following manner

$$a(k)|n_k\rangle = \sqrt{n_k} \; |n_k - 1\rangle \tag{2.11}$$

$$a^\dagger(k)|n_k\rangle = \sqrt{n_k + 1} \; |n_k + 1\rangle. \tag{2.12}$$

2.3 **Complex Klein-Gordon Field**

The complex Klein-Gordon equation occurs in those cases in which the scalar field possesses an electric charge. That which corresponds to the probability and probability current in the Schrodinger equation is a 4-vector in the Klein-Gordon equation and represents the charge and current density. The time component, ρ, is given by

$$\rho = \frac{i\hbar}{2m} \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) \tag{2.13}$$

and the 4-vector is

$$j^\mu = (\rho, \mathbf{j}) = \frac{i\hbar}{m}\phi^* \overleftrightarrow{\partial^\mu} \phi, \tag{2.14}$$

where $\phi^* \overleftrightarrow{\partial^\mu} \phi = \phi^* \partial^\mu \phi - \partial^\mu(\phi^*)\phi$. Both ρ and $\mathbf{j}$ vanish when only the real Klein-Gordon equation is considered, i.e. the scalar field possesses no electric charge (e.g. Ryder, 1996).

The formalism used in this chapter will be used throughout the rest of the dissertation but with modifications to accommodate the higher derivative aspects of the K-G equation.

CHAPTER 3

Higher-Derivative Klein-Gordon Equations in Cartesian Coordinates

3.1 Poincaré Group, Eigenvalue Equations, Elementary Particles

To derive dynamical equations for particles on a space-time manifold, we begin by defining carefully what we mean by an elementary particle. We are interested in quantum particles, objects which may be described by a function of space-time variables that we associate with a space-time, metric manifold. An elementary particle is any object that may be described by a function existing in a Hilbert space and transforming like an irreducible representation (irrep) of some symmetry group. The symmetry group may be chosen arbitrarily, defining a particular type of particle. In this thesis, we are interested in the symmetry group consisting of all the coordinate transformations leaving the space-time metric of a given manifold invariant. We limit ourselves to flat space-time and employ the Minkowski metric (Wigner, 1939; Kim and Noz, 1986).

The Minkowski metric can be written as $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, where the spatial coordinates x, y and z, and time t are all measured in the same units because the speed of light has been defined as $c = 1$. The group of this metric is the Poincaré group, whose structure is given by the following semi-direct product: $P = H_p \otimes_s T(3{+}1)$, where $T(3{+}1)$ is an invariant subgroup of space-time translations and H_p is a non-invariant subgroup consisting of the remaining transformations and the identity transformation (Kim and Noz, 1986; Weldon, 2003). In this thesis, we consider the so-called proper orthochronous group $P_+^\uparrow$ that is a subgroup of P. To identify an elementary particle, we require that a scalar, analytical wave function

$\psi(x^\mu)$ transform as one of the irreps of the invariant subgroup $T(3+1)$. Since the transformation properties of the function are preserved in the irreps of the semi-direct product of the group, it can be shown that a necessary condition that ψ represent an elementary particle in any inertial frame of reference is the following set of eigenvalue equations (Fry et al., 2011)

$$i\partial^\mu \psi \;=\; k^\mu \psi \;, \tag{3.1}$$

where $\partial^\mu = \partial/\partial x_\mu = (\partial_t, -\nabla)$, $k^\mu = (\omega, k^i)$, $k_\mu = (\omega, -k_i)$, with $\partial_0 = \partial_t = \partial/\partial t$, $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$ and $k^\mu k_\mu = \omega^2 - k^i k_i = \omega^2 - \mathbf{k}\cdot\mathbf{k} = \omega^2 - \mathbf{k}^2$. Since $i\partial^\mu$ is a Hermitian operator, the k^μ must be real numbers. The irreps of $T(3+1)$ and thus $P_+^\uparrow$ may be labelled by real numbers. By contrast, if we label the irreps by a parameter called mass by Wigner (1939), some values of this parameter may become imaginary (Kim and Noz, 1986). We will now use the set of eigenvalue equations given by equation (3.1) to derive Poincaré invariant dynamical equations in Minkowski space-time (Musielak et al., 2015).

3.2 Two Series of Higher-Derivative Equations

Let us consider two inertial observers who use sets of coordinates x^μ and x'^μ to describe the state of the particle that is given by the following two scalar functions: $\psi(x^\mu)$ and $\psi'(x'^\mu)$. In space-time with the Minkowski metric the coordinates x^μ and x'^μ are related to each other by the Lorentz transformation Λ^μ_ν, which can be used to obtain $x'^\mu = \Lambda^\mu_\nu x^\nu$. With this coordinate transformation, it is easy to show that $(\partial'^\mu \partial'_\mu)\psi'(x'^\mu) = (\partial^\mu \partial_\mu)\psi(x^\mu)$. The operator $(\partial'^\mu \partial'_\mu)$ is one of two Casimir operators for the Poincaré group (Kim and Noz, 1986) and the only one needed to develop possible higher order equations for scalar functions.

Using the above results, the simplest Poincaré invariant dynamical equation that can be derived from the eigenvalue equations is

$$-\partial^\mu \partial_\mu \psi \;=\; k^\mu k_\mu \psi \;=\; (\omega^2 - \mathbf{k}^2)\psi \;, \quad (\mu = 0, 1, 2, 3), \tag{3.2}$$

which is commonly known as the Klein-Gordon (K-G) equation (Ryder, 1996; Greiner, 1990). We can write this equation in a more compact form by introducing a special frame of reference with $k^i = 0 \;\; (i = 1, 2, 3)$, so that $\omega = \omega_0$ in that special reference frame. We call ω_0 the invariant frequency since it is the same for all inertial observers (Fry et al., 2011). This allows us to write $\omega^2 = \omega_0^2 + \mathbf{k}^2$, where $\mathbf{k}^2 = \mathbf{k}\cdot\mathbf{k} = k^i k_i$, $(i = 1, 2, 3)$, and since $\mathbf{k}^2 = 0$ then $\omega^2 = \omega_0^2$ and we obtain

$$\left(\partial^\mu \partial_\mu + \omega_0^2\right)\psi = 0 \;. \tag{3.3}$$

We note that classical mass and the Planck constant do not appear and are in fact not needed since ω_0 can be determined experimentally. In developing higher order dynamical equations for particles or fields we rely upon the fundamental equation (3.1), which is a set of eigenvalue equations defining an elementary particle. The above method used to obtain the KG equation can now be applied two times and three times to derive the following fundamental dynamical equations

$$(\partial^\mu \partial_\mu)^2 \psi - \omega_0^4 \psi = 0 \;, \tag{3.4}$$

and

$$(\partial^\mu \partial_\mu)^3 \psi + \omega_0^6 \psi = 0 \;. \tag{3.5}$$

After repeating the procedure an odd number of times or an even number of times, the resulting Poincaré invariant dynamical equations are, respectively,

$$\left[(\partial^\mu \partial_\mu)^n + \; \omega_0^{2n}\right]\psi = 0 \;, \tag{3.6}$$

and

$$\left[(\partial^\mu \partial_\mu)^m - \ \omega_0^{2m}\right] \psi = 0 \ , \tag{3.7}$$

where ω_0 is a constant, and **n** is any positive **odd** integer and **m** is any positive **even** integer. Based on our approach presented here, the above equations exhaust all possibilities of obtaining the fundamental linear equations for scalar state functions with no interactions. It must be noted that these higher order differential equations have solutions other than elementary state functions of the eigenvalue equations (equation (3.1)), possibly introducing new physical phenomena in the form of fields (possibly, for example, tachyonic fields). An interesting result obtained here is that there are two distinct sets of Poincaré invariant dynamical equations: one infinite set with the odd powers of $(\partial^\mu \partial_\mu)$ and the 'plus' sign in front of ω_0, and the other infinite set with the even powers of $(\partial^\mu \partial_\mu)$ and the 'minus' sign in front of ω_0. In the remaining parts of this thesis, we shall refer to equations (3.6) and (3.7) as the odd and even order fundamental dynamical equations, respectively (Musielak et al., 2015).

It should be noted that, in reality, the "odd" series is contained in the "even" series in the following sense: If m (m even) is equal to $2n$ (n odd), then the Klein-Gordon equation to the power $m = 2n$ can be factored as

$$[(\partial^\mu \partial_\mu)^{2n} - (\omega_0^2)^{2n}]\, \psi = [(\partial^\mu \partial_\mu)^{n} - (\omega_0^2)^{n}]\, [(\partial^\mu \partial_\mu)^{n} + (\omega_0^2)^{n}]\, \psi = 0$$

(Musielak et al., 2015).

3.3 The Odd Order Series with One Real Root

The first equation in this odd order series is for $n = 1$ and the corresponding equation is

$$\left(\partial^\mu \partial_\mu + \omega_0^2\right) \psi = 0 \ . \tag{3.8}$$

This, of course, is just the K-G equation. The higher order equations in the odd order series are of the form of equation (3.6), where, again, n is a positive odd integer,

$$\left[(\partial^\mu \partial_\mu)^n + \ \omega_0^{2n}\right] \psi = 0 \ , \tag{3.9}$$

with $\psi(t,\mathbf{x}) = \displaystyle\int \frac{d^3k}{(2\pi)^3} \phi(t,\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$ and the general form of ϕ as $\phi(t,\mathbf{k}) = A(\mathbf{k}) e^{\pm i\omega t}$. Equations of the form of equation (3.9) can be factored as

$$\left[(\partial^\mu \partial_\mu)^n + \ \omega_0^{2n}\right] \psi = \left(\partial^\mu \partial_\mu + \omega_0^2\right) \left[\sum_{k=1}^{n} (-1)^{k+1} \ (\partial^\mu \partial_\mu)^{n-k} \ \omega_0^{2(k-1)}\right] \psi = 0. \tag{3.10}$$

Equation 3.10 has two factors operating on ψ. The first factor operating on ψ is obviously the K-G equation whose solutions are real. The second factor operating on ψ provides solutions for which the ω's are complex or purely imaginary. Therefore, if we are looking for physically realistic solutions, then the higher derivative odd order series offer nothing more than the K-G equation and solutions to the K-G equation.

3.3.1 Example for $n = 3$

To make this more explicit, let's take as an example $n = 3$ which gives

$$\left[(\partial^\mu \partial_\mu)^3 + \omega_0^6\right] \psi = 0 \ . \tag{3.11}$$

The Lagrange density for this equation is

$$\mathcal{L} = \frac{1}{2} \left[(\partial^\mu)^3 \psi \, (\partial_\mu)^3 \psi - \omega_0^6 \psi^2\right] . \tag{3.12}$$

Expanding the higher derivative operator of equation (3.11) will give us

$$(\partial^\mu \partial_\mu)^3 = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)^3 = \frac{\partial^6}{\partial t^6} - 3\frac{\partial^4}{\partial t^4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$+3\frac{\partial^2}{\partial t^2}\left[2\left(\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial x^2\partial z^2}+\frac{\partial^4}{\partial y^2\partial z^2}\right)+\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}+\frac{\partial^4}{\partial z^4}\right)\right]$$

$$-3\left(\frac{\partial^6}{\partial x^4\partial y^2}+\frac{\partial^6}{\partial x^4\partial z^2}+\frac{\partial^6}{\partial x^2\partial y^4}+\frac{\partial^6}{\partial y^4\partial z^2}+\frac{\partial^6}{\partial x^2\partial z^4}+\frac{\partial^6}{\partial y^2\partial z^4}\right)$$

$$-6\ \frac{\partial^6}{\partial x^2\partial y^2\partial z^2}-\left(\frac{\partial^6}{\partial x^6}+\frac{\partial^6}{\partial y^6}+\frac{\partial^6}{\partial z^6}\right). \tag{3.13}$$

This equation can be written more succinctly using ∇^2

$$(\partial^\mu\partial_\mu)^3=\frac{\partial^6}{\partial t^6}-3\ \frac{\partial^4}{\partial t^4}\nabla^2+3\ \frac{\partial^2}{\partial t^2}(\nabla^2)^2-(\nabla^2)^3. \tag{3.14}$$

So equation (3.14) is the differential operator form that will be used in equation 3.11. The Fourier transform in space is

$$\psi(t,\mathbf{x})=\int\frac{d^3k}{(2\pi)^3}\ \phi(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.15}$$

Applying $(\partial^\mu\partial_\mu)^3+\omega_0^6$ to both sides of the Fourier transform will give us a 6th order differential equation of $\phi(t,\mathbf{k})$ with respect to time, t,

$$\frac{\partial^6\phi}{\partial t^6}+3\ k^2\ \frac{\partial^4\phi}{\partial t^4}+3\ (k^2)^2\ \frac{\partial^2\phi}{\partial t^2}+\left[(k^2)^3+\omega_0^6\right]\phi(t,\mathbf{k})=0. \tag{3.16}$$

For the solutions to equation (3.16) we take $\phi(t,\mathbf{k})=A(\mathbf{k})e^{\pm i\omega t}$, where $A(\mathbf{k})$ and ω are to be determined. The six possible solutions for ω are $\omega_{1\pm}=\pm\sqrt{k^2+\omega_0^2}$, $\omega_{2\pm}=\pm\sqrt{k^2+\omega_0^2\ e^{\frac{2\pi i}{3}}}$, and $\omega_{3\pm}=\pm\sqrt{k^2+\omega_0^2\ e^{\frac{-2\pi i}{3}}}$. Thus, using Weldon (2003), the solutions for $\phi(t,\mathbf{k})$ are

$$\phi_1(t,\mathbf{k})=\phi(t,\mathbf{k})=\sum_{r=1}^{3}\left(a_r e^{-i\omega_r t}+b_r e^{i\omega_r t}\right), \tag{3.17}$$

$$\phi_2(t,\mathbf{k}) = \frac{d\phi}{dt} = \sum_{r=1}^{3}\left(a_r(-i\omega_r)e^{-i\omega_r t} + b_r(i\omega_r)e^{i\omega_r t}\right), \tag{3.18}$$

$$\phi_3(t,\mathbf{k}) = \frac{d^2\phi}{dt^2} = \sum_{r=1}^{3}\left(a_r(-\omega_r^2)e^{-i\omega_r t} + b_r(-\omega_r^2)e^{i\omega_r t}\right). \tag{3.19}$$

The momenta π_1, π_2 and π_3 are given by

$$\pi_1 = \sum_{r=1}^{3}\left[\omega_r(3k^4 - 3k^2\omega_r^2 + \omega_r^4)\left(ia_r e^{-i\omega_r t} - ib_r e^{i\omega_r t}\right)\right] , \tag{3.20}$$

$$\pi_2 = \sum_{r=1}^{3}\left[\omega_r^2(\omega_r^2 - 3k^2)\left(a_r e^{-i\omega_r t} + b_r e^{i\omega_r t}\right)\right] , \tag{3.21}$$

$$\pi_3 = \sum_{r=1}^{3}\left[(-i\omega_r^2)\left(a_r e^{-i\omega_r t} - b_r e^{i\omega_r t}\right)\right] . \tag{3.22}$$

Again using Weldon (2003), we define an expression for this example

$$\frac{1}{R_r} = -6\omega_r(k^4 - 2k^2\omega_r^2 + \omega_r^4) \tag{3.23}$$

and define the commutators for a_r and b_s as

$$[a_r, b_s] = \delta_{rs}R_r \tag{3.24}$$

and in that case the commutation relationships obeyed by the functions ϕ_m and the momenta π_n will be

$$[\phi_m, \pi_n] = i\delta_{mn} \tag{3.25}$$

$$[\phi_m, \phi_n] = [\pi_m, \pi_n] = 0, \tag{3.26}$$

where $m = 1, 2, 3$ and $n = 1, 2, 3$. Ostrogradski, who generalized Lagrange mechanics to higher derivatives as discussed by Whittaker (1947), defined the Hamiltonian in this case to be

$$H = \sum_{r=1}^{3}\left[-3\omega_r(k^4 - 2k^2\omega_r^2 + \omega_r^4)\left(a_r b_r + b_r a_r\right)\right] . \tag{3.27}$$

or using $\frac{1}{R_r}$ becomes

$$H = \sum_{r=1}^{3} \omega_r \left(\frac{b_r a_r}{R_r} + \frac{1}{2} \right) \tag{3.28}$$

For a fixed wave vector, $\mathbf{k}$, the following commutators are satisfied

$$[H, a_r] = -\omega_r a_r, \quad [H, b_r] = \omega_r b_r, \quad [H, \phi_n] = -i\frac{d\phi_n}{dt}, \tag{3.29}$$

where $n = 1, 2, 3$.

Under the assumption that the field operators ϕ_1, ϕ_2 and ϕ_3 are self-adjoint, we get $b_r = a_r^\dagger$ and by taking into account only real and positive frequency $\omega_r \equiv \omega_{1+}$ the Hamiltonian is simply

$$H = \omega_{1+} \left(\frac{a_1^\dagger a_1}{R_1} + \frac{1}{2} \right). \tag{3.30}$$

Following Weldon (2003), we define the natural vacuum $|\text{vac}\rangle$ of the Fock space as $a_{1+}|\text{vac}\rangle = 0$ with the zero-point energy $H|\text{vac}\rangle = E_0|\text{vac}\rangle$, where $E_0 = \omega_{1+}$. For a one-particle state, we have

$$H a_{1+}^\dagger |\text{vac}\rangle = (\omega_{1+} + E_0) a_{1+}^\dagger |\text{vac}\rangle \ , \tag{3.31}$$

with the norm being given by

$$\langle \text{vac}| a_{1+} a_{1+}^\dagger |\text{vac}\rangle = R_1. \tag{3.32}$$

3.3.2 Discussion

In the example for $n = 3$ the equation is a 6th-order Poincaré invariant dynamical equation whose solutions are a consequence of the metric and correspond to

$$\omega_{1\pm} = \pm\sqrt{k^2 + \omega_0^2}, \quad \omega_{2\pm} = \pm\sqrt{k^2 + \omega_0^2 \ e^{\frac{2\pi i}{3}}}, \quad \omega_{3\pm} = \pm\sqrt{k^2 + \omega_0^2 \ e^{\frac{-2\pi i}{3}}}.$$

The requirement of the eigenvalue equations (equation (3.1)) that the solutions be real is obeyed by only one of the ω's, namely $\omega_{1\pm}$ with $\omega_0 > 0$. In fact $\omega_{2\pm}$ and $\omega_{3\pm}$ are complex conjugates, i.e. not strictly real, and are not considered as physical solutions. Moreover, $\omega_{1\pm}$ is consistent with the Special Theory of Relativity whereas $\omega_{2\pm}$ and $\omega_{3\pm}$ are not.

The results for the ω's for $n = 3$ (6th-order ODE) can be generalized to any odd $n \geq 3$. There will be, of course, an odd number of ω's but ω_1 will be the solution to the Klein-Gordon equation which will leave an even number of remaining ω's. These will all be in complex conjugate pairs which will not be real solutions to the $2n$-order differential equation and thus, physically unacceptable. The ω's will be of the form

$$\omega_{L\pm} = \pm\sqrt{k^2 + \omega_0^2 e^{\frac{i2\pi l}{L}}}, \tag{3.33}$$

where $L = 1, 2, ..., n$ and $l = 0, 1, ..., L-1$. Hence, the ω's, which represent energy, are complex quantities.

Therefore, we reach the conclusion that all the higher derivative equations of the odd series have only the solutions of the Klein-Gordon equation as physically acceptable solutions. As a result, for the odd series there are no fundamental dynamical equations that could be used to construct new higher-derivative quantum field theories.

3.4 The Even Order Series with Two Real Roots

We now consider the even order series, i.e.

$$\left[(\partial^\mu \partial_\mu)^m - \ \omega_0^{2m}\right] \psi = 0, \tag{3.34}$$

where m is a positive even integer. This general form of the operator can be factored in the following manner.

$$(\partial^\mu \partial_\mu)^m - \ \omega_0^{2m} = \left[(\partial^\mu \partial_\mu) + \omega_0^2\right] \left[(\partial^\mu \partial_\mu) - \omega_0^2\right] \left[\sum_{k=1}^{m/2} \left((\partial^\mu \partial_\mu)^{m-2k} \, \omega_0^{2(k-1)}\right)\right]. \tag{3.35}$$

This operator has three factors and when applied to the function ψ equals zero. The first two factors applied individually to ψ yield real solutions whereas the last factor applied to ψ yield only imaginary or complex solutions. As we are only concerned with physically meaningful (real) solutions, the first two factors applied to ψ are those which represent equations of interest. All of the higher derivative equations of the even series can be reduced to three factors, only two of which yield real and therefore physically meaningful solutions. What is of notable interest in the even series is that it produces the Klein-Gordon equation (first factor operating on ψ) as well as a second, different equation (second factor operating on ψ) with real solutions, unlike the odd series equations which yield only the Klein-Gordon equation for real solutions. Since all of the higher derivative equations in the even series ultimately produce the same two factors which give real solutions for ω, we need only consider the lowest order ($m = 2$) in the even series. However, higher even orders with $m \geq 4$ will give solutions of the lowest order plus additional complex conjugate solutions. The lowest order equation in this series is

$$\left[(\partial^\mu \partial_\mu)^2 - \omega_0^4\right] \psi(t, \mathbf{x}) = 0 \tag{3.36}$$

and the corresponding Lagrange density is

$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu)^2 \psi (\partial_\mu)^2 \psi - \omega_0^4 \psi^2\right]. \tag{3.37}$$

If the Euler-Lagrange equation is applied directly to the Lagrange density we should recover the higher-derivative Klein-Gordon equation. The Euler-Lagrange equation for this higher-derivative Lagrange density is of the form

$$(\partial^\mu)^2 \left(\frac{\partial \mathcal{L}}{\partial\left[(\partial^\mu)^2 \psi\right]}\right) + (\partial_\mu)^2 \left(\frac{\partial \mathcal{L}}{\partial\left[(\partial_\mu)^2 \psi\right]}\right) + \frac{\partial \mathcal{L}}{\partial \psi} = 0. \tag{3.38}$$

Evaluating this Euler-Lagrange equation leads immediately to the higher-derivative Klein-Gordon equation

$$\left[(\partial^\mu \partial_\mu)^2 - \omega_0^4\right] \psi = 0. \tag{3.39}$$

To obtain the Lagrangian, the Lagrange density is Fourier transformed in space using

$$\psi(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\,\phi(t,\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.40}$$

leading to

$$L = \frac{1}{2}\left[\left(\frac{d^2\phi}{dt^2}\right)^2 + 2k^2\frac{d^2\phi}{dt^2}\phi + k^4\phi^2 - \omega_0^4\phi^2\right]e^{2i\mathbf{k}\cdot\mathbf{x}}, \tag{3.41}$$

where $\phi = \phi(t,\mathbf{k})$, $\mathbf{k}$ is the wave 3-vector and $\mathbf{x}$ is the spatial 3-vector.

Using the Lagrange-Euler equation once again but with the Lagrangian instead of the Lagrange density

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial\ddot{\phi}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) + \frac{\partial L}{\partial\phi}$$

$$= \frac{1}{2}\left[2\frac{d^4\phi}{dt^4} + 2k^2\frac{d^2\phi}{dt^2} + 2k^2\frac{d^2\phi}{dt^2} + 2(k^4-\omega_0^4)\phi\right]e^{2i\mathbf{k}\cdot\mathbf{x}}$$

$$= \frac{1}{2}\left[2\frac{d^4\phi}{dt^4} + 4k^2\frac{d^2\phi}{dt^2} + 2(k^4-\omega_0^4)\phi\right]e^{2i\mathbf{k}\cdot\mathbf{x}}$$

$$= \left[\frac{d^4\phi}{dt^4} + 2k^2\frac{d^2\phi}{dt^2} + (k^4-\omega_0^4)\phi\right]e^{2i\mathbf{k}\cdot\mathbf{x}} = 0, \tag{3.42}$$

which finally produces a 4th order differential equation of $\phi(t,\mathbf{k})$

$$\frac{d^4\phi}{dt^4} + 2k^2\frac{d^2\phi}{dt^2} + \left(k^4-\omega_0^4\right)\phi = 0. \tag{3.43}$$

As a solution to equation (3.43) we try $\phi(t,\mathbf{k}) = A(\mathbf{k})e^{\pm i\omega t}$, where A and ω are to be determined. The solutions of $\phi(t,\mathbf{k})$ are found to be those with the following frequencies

$$\omega_{1\pm} = \pm\sqrt{k^2+\omega_0^2} \tag{3.44}$$

and

$$\omega_{2\pm} = \pm\sqrt{k^2 - \omega_0^2}. \tag{3.45}$$

Thus, we can write

$$\phi(t, \mathbf{k}) = \sum_{r=1}^{2} \left[a_r(\mathbf{k}) e^{-i\omega_r t} + b_r(\mathbf{k}) e^{i\omega_r t} \right], \tag{3.46}$$

where a_r and b_r represent A for different values of ω_r (ω_r is either ω_1 or ω_2). Of course, $\phi(t, \mathbf{k}), a_r, b_r$ are now interpreted as operators.

3.4.1 Quantization

We use the method of Weldon (2003) for the quantization of higher-derivative field theories

$$\phi_n = \frac{d^{n-1}}{dt^{n-1}\phi} \quad \text{and} \quad \pi_m = \sum_{l=m}^{N} (-1)^{l-m} C_l \frac{d^{2l-m}\phi}{dt^{2l-m}},$$

where the ϕ_n are the canonical coordinates, C_l are constants determined by the field equation, and the π_m are the canonical momenta. Since $N = 2$ there will be two coordinate fields, ϕ_1 and ϕ_2, and two canonical momenta, π_1 and π_2. The explicit forms for the coordinates and momenta are

$$\phi_1 = \phi = \sum_{r=1}^{2} \left[a_r e^{-i\omega_r t} + b_r e^{i\omega_r t} \right], \tag{3.47}$$

$$\phi_2 = \frac{d\phi}{dt} = \sum_{r=1}^{2} \left[a_r(-i\omega_r) e^{-i\omega_r t} + b_r(i\omega_r) e^{i\omega_r t} \right], \tag{3.48}$$

$$\pi_1 = \sum_{r=1}^{2} \left[\omega_r \left(-2k^2 + \omega_r^2 \right) \left(-i a_r e^{-i\omega_r t} + i b_r e^{i\omega_r t} \right) \right], \tag{3.49}$$

$$\pi_2 = -\sum_{r=1}^{2} \left[\omega_r^2 \left(a_r e^{-i\omega_r t} + b_r e^{i\omega_r t} \right) \right] \tag{3.50}$$

and they obey the following commutation relations

$$[\pi_m, \phi_n] = -i\delta_{mn} \tag{3.51}$$

$$[\phi_m, \phi_n] = [\pi_m, \pi_n] = 0, \tag{3.52}$$

where $m = 1, 2$ and $n = 1, 2$.

Again, using the method of Weldon (2003) we introduce a term which in our case with $N = 2$ will be

$$R_r = \frac{-1}{4\omega_r(k^2 - \omega_r^2)}, \tag{3.53}$$

where $r = 1, 2$. For equation (3.51) to be true the commutator of the operators a_r and b_s must satisfy

$$[a_r, b_s] = \delta_{rs} R_r. \tag{3.54}$$

Ostrogradski (Whittaker,1947) defined the Hamiltonian, H, as

$$H = -L + \pi_1 \phi_2 + \pi_2 \frac{d^2\phi_1}{dt^2} \tag{3.55}$$

which becomes

$$H = 2\sum_{r=1}^{2} \left[\omega_r^2(k^2 + \omega_r^2)(a_r b_r + b_r a_r)\right] = \sum_{r=1}^{2} \omega_r \left(\frac{b_r a_r}{R_r} + \frac{1}{2}\right). \tag{3.56}$$

For a fixed wave vector, **k**, the following commutators are satisfied

$$[H, a_r] = -\omega_r a_r, \quad [H, b_r] = \omega_r b_r, \quad [H, \phi_n] = -i\frac{d\phi_n}{dt}, \tag{3.57}$$

where $n = 1, 2$.

Let us now assume that the field operators ϕ_1 and ϕ_2 are self-adjoint, so that we have $b_r = a_r^\dagger$ with $[a_r, a_s^\dagger] = \delta_{rs} R_r$. Moreover, we consider only positive and physically acceptable (real) ω_r. Since ω_{1+} and ω_{2+} (equations (3.44) and (3.45)) are the only frequencies that satisfy these conditions, we write the Hamiltonian as

$$H = \sum_{r=1}^{2} \omega_r \left(\frac{a_r^\dagger a_r}{R_r} + \frac{1}{2}\right) , \tag{3.58}$$

where $\omega_r = \omega_{1+}, \ \omega_{2+}$, and $a_r^\dagger$ and a_r are the creation and annihilation operators corresponding to these frequencies.

Following Weldon (2003), we define the natural vacuum $|\mathrm{vac}\rangle$ of the Fock space as $a_r|\mathrm{vac}\rangle = 0$ with the zero-point energy $H|\mathrm{vac}\rangle = E_0|\mathrm{vac}\rangle$, where $E_0 = \dfrac{\omega_{1+} + \ \omega_{2+}}{2}$. For a one-particle state, we have

$$Ha_r^\dagger|\mathrm{vac}\rangle = (\omega_r + E_0)a_r^\dagger|\mathrm{vac}\rangle \ , \tag{3.59}$$

with the norm being given by

$$\langle\mathrm{vac}|a_r a_s^\dagger|\mathrm{vac}\rangle = \delta_{rs}R_r \ . \tag{3.60}$$

3.4.2 Discussion

The 4th-order ($m = 2$) Poincaré invariant, dynamical equation has solutions for which there are two ω's, i.e. $\omega_{1\pm} = \pm\sqrt{k^2 + \omega_0^2}$ and $\omega_{2\pm} = \pm\sqrt{k^2 - \omega_0^2}$, where ω_0 is real and $\omega_0 > 0$. These solutions are consequences of the (Minkowski) metric. Clearly, ω_{1+} and ω_{1-} are solutions to the Klein-Gordon equation. They describe spin-0, massive particles of ordinary matter, with ω_{1+} representing particles (positive energy) and ω_{1-} representing antiparticles (negative energy). For ordinary matter, then, $\omega_{1\pm} = \pm\sqrt{k^2 + \omega_0^2}$ which can be written as

$$\omega_{1\pm}^2 - k^2 = \omega_0^2. \tag{3.61}$$

This equation describes the "mass shell" and consists of a two-sheeted hyperboloid: one with $\omega_{1+} \geq \omega_0$ and the other with $\omega_{1-} \leq -\omega_0$. These two sets of solutions will be separate and distinct for any proper Lorentz transformation (Schwartz, 2016). See figure 3.1.

Now $\omega_{2\pm}$ are solutions to the second factor in equation (3.35) as applied to ψ, i.e. $\left[(\partial^\mu\partial_\mu) - \omega_0^2\right]\psi = 0$. This equation is the Klein-Gordon equation for tachyonic or space-like fields whose quantization gives tachyons (Schwartz, 1982, 2011, 2016). Its solutions, $\omega_{2\pm} = \pm\sqrt{k^2 - \omega_0^2}$, can be rewritten as

$$\omega_{2\pm}^2 - k^2 = -\omega_0^2 \tag{3.62}$$

and describes a mass shell which is a hyperboloid of one sheet in four dimensions. Positive and negative values of $\omega_{2\pm}$ are not separated. See figure 3.2.

The Klein-Gordon equation for tachyons or space-like fields in the higher-derivative equations occurs without any assumptions of superluminal velocities for

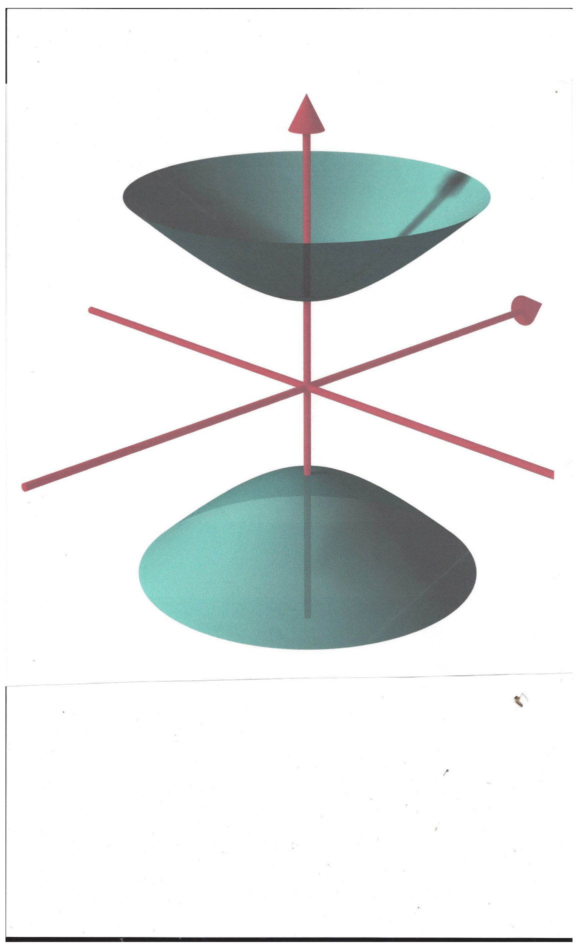

Figure 3.1 Hyperboloid of two sheets

particles, fields or reference frames. Its appearance is the result of $4n$ $(n = 1, 2, 3, ...)$ repeated applications of the eigenvalue equation (3.1), i.e. $i\partial^\mu \psi \; = \; k^\mu \psi$.

If ω_1 represents the energy, then the solutions $\omega_{1\pm} = \pm\sqrt{k^2 + \omega_0^2}$ satisfy the Special Relativity energy-momentum relationship. It would seem that $\omega_{2\pm} = \pm\sqrt{k^2 - \omega_0^2}$ would not be consistent with this principle. However, Hill and Cox (2012), have proposed transformations (in fact, two new transformations) applied to inertial reference frames with relative velocities greater than the speed of light which are, as they say, transformations "complementary" to the Lorentz transformations of Special Relativity for subluminal velocities. Further, with these transformations there is no need for

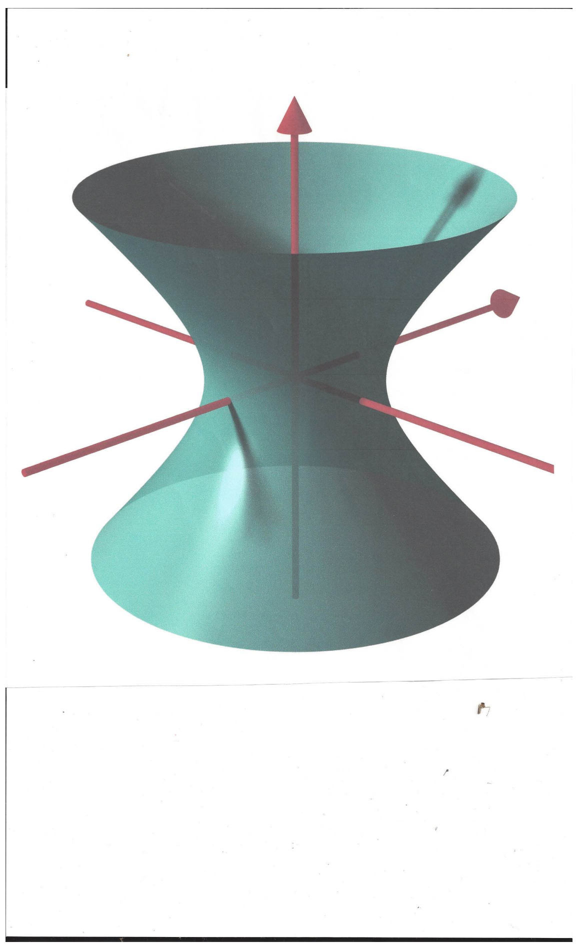

Figure 3.2 Hyperboloid of one sheet

contrived concepts such as imaginary mass nor complicated physics.

The results obtained in section 3.4 can be easily generalized to any value of m from even series, and the general solutions correspond to

$$\omega_{L\pm} = \pm\sqrt{k^2 + \omega_0^2 e^{i2l\pi/L}} \ , \tag{3.63}$$

where $L = 1, 2, 3, ...$ and $l = 0,\ 1,\ 2,\ ...\ L-1$. The solutions with $\omega_{1\pm}$ and $\omega_{2\pm}$ are obtained for $L = 1$ or $L = 2$ and $l = 0, 1$. However, for all other values of L, the frequencies $\omega_{L\pm}$ are complex conjugates, which means that they are inconsistent with our requirement that $\omega_{L\pm}$ be real.

This is an interesting result as it clearly shows that every fundamental dynamical equation of the set of infinite Poincaré invariant equations with even $m > 2$ has solutions corresponding to $\omega_{1\pm}$ and $\omega_{2\pm}$, which are the same solutions as those obtained for the fundamental dynamical equation with $m = 2$. As already stated in section 3.4, the requirement imposed by the eigenvalue equations is that the values of all ω's be real, which means that the complex conjugate solutions cannot be considered in the approach presented here. Hence, we have reached an important conclusion that among the infinite set of fundamental dynamical equations with the even values of m, the invariant equation with $m = 2$ should be preferentially used in constructing higher-derivative quantum field theories.

The explicit form of the fundamental dynamical equation with $m = 2$ is equation (3.36) above, i.e.

$$(\partial^\mu \partial_\mu)^2 \psi - \omega_0^4 \psi = 0 \ . \tag{3.64}$$

The fact that the solutions with $\omega_{1\pm}$ to this equation are the same as those given by the original Klein-Gordon equation should not be surprising because our results presented in section 3.4 established the relationship between both equations. On the other hand, the above equation includes more than just the Klein-Gordon equation and, as a result, equation (3.64) also allows for tachyonic solutions with $\omega_{2\pm}$. These solutions are relevant to the faster than the speed of light Special Relativity (Hill and Cox, 2012). They may also be of interest in string theories and in some recent attempts to explain the nature of dark energy (e.g. Bagla et al., 2003) as well as a possible addition to Hawking radiation from Schwarzschild black holes (discussed in chapter 5). Another interpretation of these solutions ($\omega_{2\pm}$) is that the wave function for equation (3.64) comprises a time-like dynamic ($\omega_{1\pm}$ for the Klein-Gordon equation) and a space-like dynamic ($\omega_{2\pm}$ for the "negative" Klein-Gordon equation). This interpretation of the solutions may offer a possible explanation for nonlocalities of quantum mechanics (discussed in chapter 6).

CHAPTER 4

Quantum Field Theory for Free Spin-0 Tachyons

Substantial work on tachyons and tachyonic fields has been done by Schwartz (1982, 2011, 2016). As he points out, for ordinary particles or fields some initial solution is contained in some finite volume and the solution at a later time will also be contained in a finite volume since the particle or field can travel no faster than the speed of light. However, tachyons can travel at arbitrarily fast speeds and therefore the idea of the solution being contained in a given finite volume does not necessarily hold. For tachyons an alternative arrangement could be a 2-dimensional surface in 3-dimensional space over all time. This alternative system means that a tachyonic particle will pass through this surface in a finite time and will also pass through a parallel surface a finite distance away in a finite time. In what follows we consider the surface to be the xy-plane (z = 0).

4.1 Solutions of K-G equation

The solutions, $\psi(\mathbf{x}, t)$, to the Klein-Gordon equation are plane waves of the form

$$\psi(x) = \psi(\mathbf{x}, t) = e^{-ip_\mu p^\mu} = e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}. \tag{4.1}$$

For ordinary matter

$$p_\mu p^\mu = E^2 - \mathbf{p}\cdot\mathbf{p} = E^2 - p^2 = m^2 \tag{4.2}$$

or rewriting the equation

$$E^2 - m^2 = p^2, \tag{4.3}$$

which shows that $E \geq m$ or $E \leq -m$.

However, for tachyonic matter

$$p_\mu p^\mu = E^2 - \mathbf{p} \cdot \mathbf{p} = E^2 - p^2 = -m^2 \tag{4.4}$$

or

$$E^2 + m^2 = p^2 \tag{4.5}$$

and therefore E can have any finite value.

4.2 Conserved current density and orthogonality

In order to talk about causal commutators for ordinary matter and for tachyons we need to consider a general superposition of plane waves using an orthogonality property derived from a conserved current. Let

$$j^\mu_{1,2} = i\psi_1^*(x)\overleftrightarrow{\partial^\mu}\psi_2(x) = i\left[\psi_1^*(x)\partial^\mu\psi_2(x) - (\partial^\mu\psi_1^*(x))\psi_2(x)\right] \tag{4.6}$$

$$\partial_\mu j^\mu_{1,2}(x) = 0 \tag{4.7}$$

be a generalized conserved current density for any two solutions, $\psi_1(x)$ and $\psi_2(x)$, of the Klein-Gordon equation. If we integrate this local conservation law over a 3-dimensional volume containing all the solutions we will have

$$\frac{d}{dt}\int d^3x\ j^0_{1,2}(\mathbf{x}, t) = 0. \tag{4.8}$$

Explicitly substituting the two solutions into this integral will give

$$\frac{d}{dt}\int d^3x\ i[\psi_1^*(x)\partial^0\psi_2(x) - (\partial^0\psi_1^*(x))\psi_2(x)]$$

$$= \frac{d}{dt}\int d^3x\ i\{e^{-i(\mathbf{p}_1\cdot\mathbf{x}-E_1t)}\partial^0 e^{i(\mathbf{p}_2\cdot\mathbf{x}-E_2t)} - [\partial^0 e^{-i(\mathbf{p}_1\cdot\mathbf{x}-E_1t)}]e^{i(\mathbf{p}_2\cdot\mathbf{x}-E_2t)}\}$$

$$= \frac{d}{dt}\int d^3x\ 2(E_1+E_2)\{e^{-i(\mathbf{p}_1\cdot\mathbf{x}-E_1t)}\ e^{i(\mathbf{p}_2\cdot\mathbf{x}-E_2t)}\}$$
$$= \frac{d}{dt}\int d^3x\ 2(E_1+E_2)\{e^{-i((\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{x}-(E_1-E_2)t)}\}$$
$$= \frac{d}{dt}\int d^3x\ 2(E_1+E_2)\{e^{-i(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{x}}\ e^{i(E_1-E_2)t}\}$$
$$= \frac{d}{dt}\left[2(E_1+E_2)\ \delta^3(\mathbf{p}_1-\mathbf{p}_2)\ e^{i(E_1-E_2)t}\right] = 0. \tag{4.9}$$

For the integral itself to be non-zero $\mathbf{p}_1$ must equal $\mathbf{p}_2$. Then the derivative of the integral is equal to zero if $E_1 = E_2$ (integral no longer a function of t) or if $E_1 = -E_2$ (integral vanishes), and this represents the orthogonality of positive and negative energy solutions of the Klein-Gordon equation.

4.3 Causal Commutators for Ordinary Matter

As noted in section 1.5 the Klein-Gordon equation in Cartesian coordinates is

$$[{\partial_t}^2 - \nabla^2 + m^2]\phi(x) = 0 \tag{4.10}$$

whose general solutions in the case of a complex field are

$$\phi(x) = \int \frac{d^3k}{\sqrt{\omega(2\pi)^3}}[a(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + b^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}] \tag{4.11}$$

$$\phi^\dagger(x) = \int \frac{d^3k}{\sqrt{\omega(2\pi)^3}}[b(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + a^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}], \tag{4.12}$$

where $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$, $b(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ are coefficients representing annihilation or creation operators. These operators obey the following equal-time commutator relations (ETCR):

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k}-\mathbf{k}') \tag{4.13}$$

$$[b(\mathbf{k}), b^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k}-\mathbf{k}'). \tag{4.14}$$

All other commutators vanish. We now calculate the commutator of the field with its adjoint at two different space-time points.

$$[\phi(x), \phi^\dagger(x')] = \int \frac{d^3k}{\sqrt{\omega(2\pi)^3}} \int \frac{d^3k'}{\sqrt{\omega(2\pi)^3}}$$

$$\left[a(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + b^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \left[b(\mathbf{k}')e^{i(\mathbf{k}'\cdot\mathbf{x}'-\omega t')} + a^\dagger(\mathbf{k}')e^{-i(\mathbf{k}'\cdot\mathbf{x}'-\omega t')}\right]$$

$$- \left[b(\mathbf{k}')e^{i(\mathbf{k}'\cdot\mathbf{x}'-\omega t')} + a^\dagger(\mathbf{k}')e^{-i(\mathbf{k}'\cdot\mathbf{x}'-\omega t')}\right] \left[a(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + b^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right].$$

Carrying out the multiplications and using equations (4.13) and (4.14) we arrive at

$$[\phi(x), \phi^\dagger(x')] = \int \frac{d^3k}{\omega(2\pi)^3} \left\{e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')]} - e^{-i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')]}\right\}$$

$$= \int \frac{d^3k}{\omega(2\pi)^3} \; (2i) \; \sin[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - \omega(t-t')]$$

$$= 2i\int \frac{d^3k}{\omega(2\pi)^3} \; \{\sin[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')] \; \cos[\omega(t-t')] - \cos[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')] \; \sin[\omega(t-t')]\}. \tag{4.15}$$

The first term under the integral integrates to zero since the sine function is an odd function and the limits of integration are understood to be from $-\infty$ to $+\infty$, and we are left with

$$[\phi(x), \phi^\dagger(x')] = \frac{-i}{4\pi^3} \int d^3k \; \cos(\mathbf{k}\cdot\mathbf{r}) \; \frac{\sin[\omega(t-t')]}{\omega}, \tag{4.16}$$

where $\mathbf{r} = (\mathbf{x}-\mathbf{x}')$. Now to the integrand in equation (4.16) we can add a term which integrates to zero, i.e. $-i\sin(\mathbf{k}\cdot\mathbf{r})\frac{\sin[\omega(t-t')]}{\omega}$ and the integral then becomes

$$[\phi(x), \phi^\dagger(x')] = \frac{-i}{4\pi^3} \int \frac{d^3k}{\omega} \; [\cos(\mathbf{k}\cdot\mathbf{r}) \; \sin[\omega(t-t')] + i\sin(\mathbf{k}\cdot\mathbf{r})\sin[\omega(t-t')] \;]$$

$$= \frac{-i}{4\pi^3} \int \frac{d^3k}{\omega} \; [\cos(\mathbf{k}\cdot\mathbf{r}) \; + i\sin(\mathbf{k}\cdot\mathbf{r})]\sin[\omega(t-t')]$$

$$= \frac{-i}{4\pi^3} \int d^3k \;\; e^{i\mathbf{k}\cdot\mathbf{r}} \;\; \frac{\sin[\omega(t-t')]}{\omega}.$$

Let $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$ and write the integral using spherical coordinates (for $\mathbf{k}$-space) so that

$$[\phi(x), \phi^\dagger(x')] = \frac{-i}{4\pi^3} \int d^3k \;\; e^{ikr\cos\theta} \;\; \frac{\sin[\omega(t-t')]}{\omega}$$

$$= \frac{-i}{4\pi^3} \int_0^\infty k^2 \; dk \int_0^{2\pi} d\phi \int_0^\pi \sin\theta \; e^{ikr\cos\theta} \; d\theta \;\; \frac{\sin[\omega(t-t')]}{\omega}.$$

After carrying out the ϕ and θ integrations we get

$$[\phi(x), \phi^\dagger(x')] = \frac{-i}{4\pi^3} \int_0^\infty k^2 \; dk \; (2\pi) \; \frac{e^{ikr} - e^{-ikr}}{ikr} \; \frac{\sin[\omega(t-t')]}{\omega}$$

$$= \frac{-i}{2\pi^2} \int_0^\infty k^2 \; dk \; \frac{2\sin kr}{kr} \; \frac{\sin[\omega(t-t')]}{\omega}$$

$$= \frac{i}{\pi^2} \; \frac{1}{r}\frac{d}{dr} \; \int_0^\infty dk \;\; \cos(kr) \; \frac{\sin[\omega(t-t')]}{\omega}. \qquad (4.17)$$

This last integral is causal, i.e. it vanishes if $|\mathbf{r}| = r > |t-t'|$ (Schwartz, 2016).

4.4 Causal Commutators for Tachyon Fields

The Klein-Gordon equation for tachyons has a $-m^2$ in the place of $+m^2$ and is therefore

$$[\partial_t{}^2 - \nabla^2 - m^2]\phi(x) = 0. \tag{4.18}$$

Since the mass shell for tachyons is different from that for ordinary matter the wave functions will use different variables representing the different geometrical structure for describing tachyonic solutions. For complex tachyons

$$\phi(x) = \int_{-\infty}^{\infty} d\omega \frac{\sqrt{k}}{\sqrt{(2\pi)^3}} \int d^2\hat{k}\ e^{i(k\hat{k}\cdot\mathbf{x}-\omega t)} a(\omega,\hat{k}), \quad k = +\sqrt{\omega^2+m^2}, \tag{4.19}$$

$$\phi^\dagger(x') = \int_{-\infty}^{\infty} d\omega' \frac{\sqrt{k'}}{\sqrt{(2\pi)^3}} \int d^2\hat{k}'\ e^{-i(k'\hat{k}'\cdot\mathbf{x}'-\omega' t')} a^\dagger(\omega',\hat{k}'), \tag{4.20}$$

where $\mathbf{k}$ ($\mathbf{k}'$) is split into its magnitude k (k') and its direction $\hat{k}$ ($\hat{k}'$).
The commutator between $a(\omega,\hat{k})$ and $a^\dagger(\omega',\hat{k}')$ is postulated to be

$$[a(\omega,\hat{k}), a^\dagger(\omega',\hat{k}')] = \delta(\omega-\omega')\ \delta^2(\hat{k}-\hat{k}')\ \eta\cdot\hat{k}. \tag{4.21}$$

The vector η is chosen to be orthogonal to some reference plane (x-y plane) for the quantization we are after. In the case of tachyons we could choose $\eta = (0,0,0,1)$.
Now the commutator relation between the fields will be

$$\begin{aligned}[\phi(x),\phi^\dagger(x')] &= \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)^3}\ e^{-i\omega(t-t')} \int d^2\hat{k}\ e^{ik\hat{k}\cdot(\mathbf{x}-\mathbf{x}')}\ k\ \eta\cdot\hat{k} \\ &= -i\frac{4\pi}{(2\pi)^3}\ \eta\cdot\nabla \int_{-\infty}^{\infty} d\omega\ e^{-i\omega(t-t')}\ \frac{\sin(kr)}{kr},\end{aligned} \tag{4.22}$$

where $r = |\mathbf{x}-\mathbf{x}'|$. This integral is causal for tachyons and vanishes for $|t-t'| > r$ (Schwartz, 2016).

4.5 Charge and Number Operators

For ordinary matter the charge, Q, of the scalar field is a conserved quantity and is given by

$$Q = \int j^0(x)\ d^3x = i \int [\phi^\dagger(x)\ \partial^0 \phi(x) - (\partial^0 \phi^\dagger(x))\ \phi(x)]\ d^3x. \qquad (4.23)$$

Inserting the explicit form of the field operators and using normal ordering we have

$$Q = i \int :\phi^\dagger(x)\ \partial^0\phi(x) - (\partial^0\phi^\dagger(x))\ \phi(x):\ d^3x$$

$$= \int \frac{d^3k}{(2\pi)^3\ 2\omega_k} [a^\dagger(k) a(k) - b^\dagger(k) b(k)] = \int \frac{d^3k}{(2\pi)^3\ 2\omega_k} [N_a(k) - N_b(k)], \quad (4.24)$$

where $a(k),\ a^\dagger(k)$ are respectively annihilation and creation operators for matter and $b(k),\ b^\dagger(k)$ are similarly for antimatter, and $N_a(k),\ N_b(k)$ are the number operators for particles and antiparticles respectively.

In the case of tachyons the charge for a scalar field will be

$$Q_\eta = \int dt\ d^2x_\perp\ \eta \cdot j = \int_{-\infty}^{\infty} d\omega \int d^2\hat{k}\ \eta \cdot \hat{k}\ a^\dagger(\omega, \hat{k})\ a(\omega, \hat{k}). \qquad (4.25)$$

It appears that $a^\dagger\ a$ would be interpreted as the number operator, i.e. the number of particles per unit interval of frequency/energy per unit of solid angle in the direction of momentum.

CHAPTER 5

Hawking Radiation from Schwarzschild Black Holes

5.1 Hawking Radiation for Massless Fields

5.1.1 Classical considerations

Consider a widely dispersed mass of low density at early times so that spacetime is almost flat (Minkowski space). Imagine that this mass collapses in a finite proper time to form a black hole. Further, suppose this black hole is a Schwarzschild black hole (no angular momentum, no charge) (Hawking 1975). The line element is

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \qquad (5.1)$$

where we let $G = c = 1$.

The outgoing radial null geodesics generate the event horizon at $r = 2M$. The apparent singularity at $r = 2M$ is not a physical singularity and can be eliminated by using other coordinates.

As described by Hawking (1975) the massless quanta are generated just before the collapsing mass forms an event horizon and becomes a black hole. The description of these quanta at late times is determined by null geodesics at early times that move inward, pass through the collapsing body and emerge as radial outgoing geodesics that reach $\mathcal{I}^+$ (future null infinity) at arbitrarily late times.

We can affinely parameterize the geodesic $x^\mu = x^\mu(\lambda)$ such that the geodesic path is a linear function of λ. Then $\frac{D}{d\lambda}(\frac{dx^\mu}{d\lambda}) = 0$, where $\frac{D}{d\lambda}$ is the covariant derivative with respect to λ. Taking the Lagrangian to be

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}, \qquad (5.2)$$

then the geodesic between points a and b is found by using the variational method

$$I = \int_a^b \mathcal{L} d\lambda. \tag{5.3}$$

If $g_{\mu\nu}$ (and therefore $\mathcal{L}$) is independent of a coordinate x^μ then, using the Euler-Lagrange equation, the conjugate momentum will be a constant for this geodesic, i.e.

$$p_\mu = \frac{\partial \mathcal{L}}{\partial(\frac{dx^\mu}{d\lambda})} = g_{\mu\nu}\frac{dx^\nu}{d\lambda} = \text{constant}. \tag{5.4}$$

The Schwarzschild metric depends only on r and θ which implies that p_t and p_ϕ are constant. So, in the plane with $\theta = \frac{\pi}{2}$,

$$p_t = g_{00}\frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = E \tag{5.5}$$

$$p_\phi = g_{33}\frac{d\phi}{d\lambda} = r^2\frac{d\phi}{d\lambda} = L. \tag{5.6}$$

For null geodesics equation (5.2) is equal to zero and

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) = E^2. \tag{5.7}$$

For radial geodesics $L = 0$ so $\dfrac{dr}{d\phi} = 0$ and

$$\frac{dr}{d\lambda} = \pm E, \tag{5.8}$$

with + sign for outgoing geodesics and - sign for incoming geodesics. From equations (5.5) and (5.8) we get (Parker and Toms 2009)

$$\frac{dt}{d\lambda} \mp \left(1 - \frac{2M}{r}\right)^{-1}\frac{dr}{d\lambda} = 0. \tag{5.9}$$

If we define $r^* = r + 2M\ln(r - 2M)$ then

$$\frac{dr^*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1} \tag{5.10}$$

and

$$\frac{d}{d\lambda}(t \mp r^*) = 0. \tag{5.11}$$

As $r \to 2M$ from the right $r^* \to -\infty$. The null coordinate

$$u = t - r^* \tag{5.12}$$

is for outgoing radial geodesics and the null coordinate

$$v = t + r^* \tag{5.13}$$

is for incoming radial geodesics. By equation (5.11) each is equal to a constant.

Given C, an incoming radial null geodesic, then $v = v_1$ for some $v_1 = \textit{constant}$ corresponding to C. C passes through the event horizon of the Schwarzschild black hole with λ an affine parameter along C. The null coordinate u along C is given by $u(\lambda)$, the form of which just exterior to the event horizon will determine the spectrum of particles created by the black hole. Along C

$$\frac{du}{d\lambda} = \frac{dt}{d\lambda} - \frac{dr^*}{d\lambda} = \frac{dt}{d\lambda} - \frac{dr^*}{dr}\frac{dr}{d\lambda} = 2\left(1 - \frac{2M}{r}\right)^{-1} E. \tag{5.14}$$

Since $\dfrac{dr}{d\lambda} = -E$ along C, after integrating we get

$$r - 2M = -E\lambda, \tag{5.15}$$

where $\lambda = 0$ for $r = 2M$. Note that for $r > 2M$, $\lambda < 0$. From equation (5.15)

$$\left(1 - \frac{2M}{r}\right)^{-1} = 1 - \frac{2M}{E\lambda} \tag{5.16}$$

and therefore, from equation (5.14)

$$\frac{du}{d\lambda} = 2E - \frac{4M}{\lambda}. \tag{5.17}$$

Integrating this equation we then obtain on the incoming null geodesic C

$$u = 2E\lambda - 4M \ln\left(\frac{\lambda}{K_1}\right), \tag{5.18}$$

where K_1 is a negative constant. At great distance from the event horizon $u \approx 2E\lambda$, whereas near the event horizon

$$u \approx -4M \ln\left(\frac{\lambda}{K_1}\right) \tag{5.19}$$

($\lambda = 0$ at the event horizon). Note that $u = -\infty$ on $\mathcal{I}^-$, the past null infinity, and $u = +\infty$ at the event horizon.

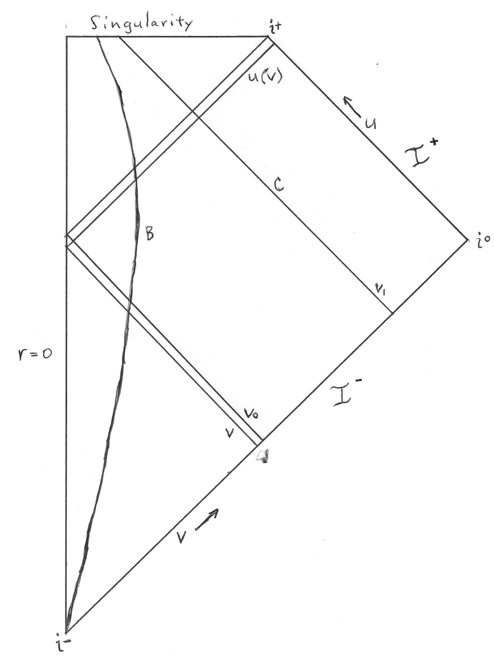

Figure 5.1 Penrose Diagram where i^+ is future timelike infinity, i^- is past timelike infinity, i^0 is spacelike infinity. See text for other symbol definitions.

As Parker explains, since $\mathcal{I}^-$ is far from the collapsing body, then the coordinate v is an affine parameter along $\mathcal{I}^-$ related to λ such that $v_0 - v = K_2\lambda$, where K_2 is a negative constant and v_0 is the last incoming ray that passes through the body and reaches $\mathcal{I}^+$. Therefore,

$$u(v) = -4M \ln\left(\frac{\lambda}{K_1}\right) = -4M \ln\left(\frac{v_0 - v}{K_1 K_2}\right), \tag{5.20}$$

$(K_1 K_2 > 0)$. This relation determines the spectrum of created particles by the black hole that will be observed at late times.

5.1.2 Quantum aspects

The massless quantized Hermitian scalar, ϕ, satisfies the Klein-Gordon equation (Hawking 1975)

$$(-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\phi] = 0 \tag{5.21}$$

or

$$\partial^\mu\partial_\mu\phi = 0 \tag{5.22}$$

in the case of the Minkowski metric.

The particles observed at late times are created a short affine distance outside the event horizon. The field of the entire spacetime exterior to the collapsing body forming the Schwarzschild black hole can be written as (Parker and Toms 2009)

$$\phi = \int d\omega(a_\omega f_\omega + a_\omega^\dagger f_\omega^*), \tag{5.23}$$

where the f_ω and f_ω^* are a complete set of solutions of equation (5.23) and satisfy the normalization

$$(f_{\omega_1}, f_{\omega_2}) = \delta(\omega_1 - \omega_2). \tag{5.24}$$

The operators a_ω and $a_\omega^\dagger$ are time-independent and obey the commutation relations

$$[a_{\omega_1}, a_{\omega_2}^\dagger] = \delta(\omega_1 - \omega_2) \tag{5.25}$$

$$[a_{\omega_1}, a_{\omega_2}] = [a_{\omega_1}^\dagger, a_{\omega_2}^\dagger] = 0, \tag{5.26}$$

where the a_ω are annihilation operators and the $a_\omega^\dagger$ are creation operators.

Inertial observers at early times and great distance from the collapsing body detect physical particles with well-defined positive frequency solutions of the wave equation. Choose f_ω to form a complete set of incoming positive frequency solutions with energy ω at early times and large distances. On $\mathcal{I}^-$ the asymptotic form is

$$f_\omega \sim \omega^{-\frac{1}{2}} r^{-1} e^{-i\omega v} Y(\theta, \phi), \tag{5.27}$$

where $v = t + r$ is the incoming null coordinate (quantum numbers l, n on Y suppressed). The operator a_ω is then the annihilation operator.

By making the expansion of the field in terms of positive frequencies at late times we will be able to calculate the spectrum of particles created. In the Schwarzschild black hole spacetime the solutions of equation (5.21) are uniquely determined by giving data on both the event horizon and on $\mathcal{I}^+$, the future null infinity, since $\mathcal{I}^+$ is not a Cauchy surface.

On $\mathcal{I}^+$ as on $\mathcal{I}^-$, positive frequency solutions are well-defined (inertial observer in Minkowski space). Let p_ω be solutions to equation (5.21) which have no Cauchy data on the event horizon and which are asymptotically outgoing with positive frequency on $\mathcal{I}^+$. Also, let p_ω and p_ω^* form a complete set of solutions on $\mathcal{I}^+$ as well as satisfying the normalization condition

$$(p_{\omega_1}, p_{\omega_2}) = \delta(\omega_1 - \omega_2). \tag{5.28}$$

The asymptotic form will be

$$p_\omega \sim \omega^{-\frac{1}{2}} r^{-1} e^{-i\omega u} Y(\theta, \phi), \tag{5.29}$$

where $u = t - r$ is the outgoing null coordinate on $\mathcal{I}^+$. A superposition of the p_ω forms an outgoing and localized wave packet at late times and large r.

As noted p_ω has no Cauchy data on the event horizon so the most general solution must also include a set of solutions, q_ω, with modes defined on the event horizon. The q_ω will have zero Cauchy data on $\mathcal{I}^+$. The form of the q_ω will not be the same as that of the p_ω since the q_ω will be solutions near the event horizon which is not Minkowski, but we will not need their form for calculating the spectrum of particles on $\mathcal{I}^+$. Let the q_ω and q^*_ω form a complete set of solutions on the horizon with normalization

$$(q_{\omega_1}, q_{\omega_2}) = \delta(\omega_1 - \omega_2). \tag{5.30}$$

At late times the p_ω and q_ω are solutions in disjoint regions and therefore

$$(q_{\omega_1}, p_{\omega_2}) = 0. \tag{5.31}$$

Also, $(q_{\omega_1}, q^*_{\omega_2}) = (q_{\omega_1}, p^*_{\omega_2}) = (p_{\omega_1}, p^*_{\omega_2}) = 0$. So we can then expand ϕ in the entire spacetime as

$$\phi = \int d\omega (b_\omega p_\omega + b^\dagger_\omega p^*_\omega + c_\omega q_\omega + c^\dagger_\omega q^*_\omega), \tag{5.32}$$

where the b_ω are annihilation operators for particles outgoing at late times at infinity. The commutation relations are

$$[b_{\omega_1}, b^\dagger_{\omega_2}] = \delta(\omega_1 - \omega_2), \qquad [c_{\omega_1}, c^\dagger_{\omega_2}] = \delta(\omega_1 - \omega_2). \tag{5.33}$$

All other commutation relations between b_{ω_1} and c_{ω_2} are zero.

Let the state vector, $|0\rangle$, be chosen to have no particles of the field incoming from $\mathcal{I}^-$, where we are in the Heisenberg picture and $|0\rangle$ is independent of time. Then

$$a_\omega |0\rangle = 0. \tag{5.34}$$

The outgoing particles form a spectrum determined by the Bogoliubov transformation giving b_ω in terms of $a_{\omega'}$ and $a^\dagger_{\omega'}$ (Fulling, 1989; Parker and Toms, 2009). Since the f_ω and the f^*_ω are a complete set for any solution, then we can write

$$p_\omega = \int d\omega' (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}), \tag{5.35}$$

where $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are complex numbers independent of the coordinates. They are the Bogoliubov coefficients.

The relations below are noted from Parker and Toms (2009).

$$b_\omega = (p_\omega, \phi) \tag{5.36}$$

$$b_\omega = \int d\omega' (\alpha^*_{\omega\omega'} a_{\omega'} - \beta^*_{\omega\omega'} a^\dagger_{\omega'}) \tag{5.37}$$

$$(p_{\omega_1}, p_{\omega_2}) = \int d\omega' (\alpha^*_{\omega_1\omega'} \alpha_{\omega_2\omega'} - \beta^*_{\omega_1\omega'} \beta_{\omega_2\omega'}) \tag{5.38}$$

$$\beta_{\omega\omega'} = -(f^*_{\omega'}, p_\omega) \tag{5.39}$$

$$\alpha_{\omega\omega'} = (f_{\omega'}, p_\omega). \tag{5.40}$$

Form a wave packet from p_ω in a frequency range around ω. The components of p_ω in terms of $f_{\omega'}$ and $f^*_{\omega'}$ can be found from equation (5.35). This wave packet approaches $\mathcal{I}^+$ along an outgoing (from the event horizon) null geodesic characterized by a large value for u. Now we follow this wave packet backward in time. Part of this wave packet will be scattered by the curved geometry near the black hole and will reach $\mathcal{I}^-$ with frequency near the original frequency ω. Another part of this wave packet will pass through the collapsing body and reach $\mathcal{I}^-$ as a superposition of the $f_{\omega'}$ and $f^*_{\omega'}$ and ω' will be highly blueshifted ($\omega' \gg \omega$).

As a result, the non-scattered part of p_ω can also be expressed in terms of $f_{\omega'}$ and $f^*_{\omega'}$ by using equation (5.35) with coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ and $\omega' \gg \omega$. As the values of ω' become arbitrarily large at late times ($u \to \infty$), the late time spectrum of outgoing particles will be determined by the asymptotic form of $\beta_{\omega\omega'}$.

Trace this second part of p_ω back along the outgoing geodesic having a very large value for u. As this wave packet passes through the collapsing body it emerges on an incoming geodesic whose value, v, is very close to v_0, the last incoming ray to pass through the collapsing body and reach $\mathcal{I}^+$. The value u is related to v and v_0 by equation (5.20)

$$u(v) = -4M \ln\left(\frac{v_0 - v}{K}\right), \tag{5.41}$$

where $K = K_1 K_2 > 0$ characterizes the affine parameter near $\mathcal{I}^+$ and $\mathcal{I}^-$. The center of this wave packet for a small range of frequencies near ω and asymptotic form of p_ω near $\mathcal{I}^+$ is determined by the principle of stationary phase. Therefore, for this part of the wave packet, the asymptotic form for p_ω is

$$p_\omega \sim \omega^{-\frac{1}{2}} r^{-1} e^{-i\omega u} Y(\theta, \phi). \tag{5.42}$$

The asymptotic form for $f_{\omega'}$ in the expansion of p_ω has the form given in equation (5.27) with $v < v_0$ ($v > v_0$ are rays that enter the black hole).

The Bogolubov coefficients are related to f_ω and p_ω by

$$\alpha_{\omega\omega'} = (f_{\omega'}, p_\omega) = C \int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{i\omega' v} e^{-i\omega u} \tag{5.43}$$

$$\beta_{\omega\omega'} = -(f^*_{\omega'}, p_\omega) = C \int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{-i\omega' v} e^{-i\omega u}, \tag{5.44}$$

where C is a constant.

Following Parker and Toms (2009), for null geodesic coordinates u and v with an affine parameter, u can be represented as a function of v, i.e.

$$u(v) = -4M \ln\left(\frac{v_0 - v}{K}\right), \tag{5.45}$$

where K is a positive constant. Now substitute for $u(v)$ and let $s \equiv v_0 - v$ in equation (5.43) and $s \equiv v - v_0$ in equation (5.44). Then

$$\alpha_{\omega\omega'} = C\int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{i\omega' v} e^{i\omega 4M \ln(\frac{v_0 - v}{K})}$$
$$= -C\int_{\infty}^{0} ds \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{-i\omega' s} e^{i\omega' v_0} e^{i\omega 4M \ln(\frac{s}{K})} \quad (5.46)$$

$$\beta_{\omega\omega'} = C\int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{-i\omega' v} e^{i\omega 4M \ln(\frac{v_0 - v}{K})}$$
$$= C\int_{-\infty}^{0} ds \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{-i\omega' s} e^{-i\omega' v_0} e^{i\omega 4M \ln(-\frac{s}{K})}. \quad (5.47)$$

Continuing to follow Parker and Toms (2009) $\alpha_{\omega\omega'}$ can be integrated along the positive s-axis, a quarter circle at infinity around the 4th quadrant and then along the imaginary axis from $-i\infty$ to 0. The $\beta_{\omega\omega'}$ integral's contour will be along the real axis, a quarter circle at infinity around the 3rd quadrant and then along the imaginary axis from $-i\infty$ to 0. In both cases the integral along the imaginary axis is equal to that along the real axis, so we substitute $s \equiv is'$. Therefore, we have

$$\alpha_{\omega\omega'} = -iC\int_{-\infty}^{0} ds' \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{\omega' s'} e^{i\omega' v_0} e^{i\omega 4M \ln(\frac{is'}{K})} \quad (5.48)$$

$$\beta_{\omega\omega'} = iC\int_{-\infty}^{0} ds' \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{\omega' s'} e^{-i\omega' v_0} e^{i\omega 4M \ln(\frac{-is'}{K})}. \quad (5.49)$$

Taking a branch cut along the negative real axis to make the logarithm single-valued and noting that $s' < 0$ we have

$$\ln\left(\frac{is'}{K}\right) = \ln\left(\frac{-i|s'|}{K}\right) = -i\frac{\pi}{2} + \ln\left(\frac{|s'|}{K}\right) \quad (5.50)$$

and

$$\ln\left(\frac{-is'}{K}\right) = \ln\left(\frac{i|s'|}{K}\right) = i\frac{\pi}{2} + \ln\left(\frac{|s'|}{K}\right). \quad (5.51)$$

Then

$$\alpha_{\omega\omega'} = -iCe^{i\omega' v_0}e^{2\pi\omega M}\int_{-\infty}^{0} ds' \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{\omega' s'}e^{i\omega 4M\ln(\frac{|s'|}{K})}, \tag{5.52}$$

and

$$\beta_{\omega\omega'} = iCe^{-i\omega' v_0}e^{-2\pi\omega M}\int_{-\infty}^{0} ds' \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{\omega' s'}e^{i\omega 4M\ln(\frac{|s'|}{K})}. \tag{5.53}$$

Thus, after squaring the absolute value of $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ and dividing α by β we find that

$$|\alpha_{\omega\omega'}|^2 = e^{8\pi M\omega}|\beta_{\omega\omega'}|^2. \tag{5.54}$$

This result is for the wave packet traced back in time through the collapsing body just before the event horizon formed (Hawking, 1975).

The wave packet, p_ω, that reaches $\mathcal{I}^+$ is formed in two parts. Tracing p_ω backward in time, one part, $p_\omega^{(1)}$, is scattered by the spacetime around the collapsing body and reaches $\mathcal{I}^-$ with the same frequency, ω, that it had on $\mathcal{I}^+$. The other part, $p_\omega^{(2)}$, passes through the collapsing body and reaches $\mathcal{I}^-$. The two parts are disjoint regions on $\mathcal{I}^-$, $v > v_0$ for $p_\omega^{(1)}$ and $v < v_0$ for $p_\omega^{(2)}$. So $p_\omega = p_\omega^{(1)} + p_\omega^{(2)}$ and

$$(p_{\omega_1}, p_{\omega_2}) = (p_{\omega_1}^{(1)}, p_{\omega_2}^{(1)}) + (p_{\omega_1}^{(2)}, p_{\omega_2}^{(2)}). \tag{5.55}$$

If $\Gamma(\omega_1)$ is the fraction of p_{ω_1} corresponding to $p_{\omega_1}^{(2)}$, then $1-\Gamma(\omega_1)$ is the fraction corresponding to $p_{\omega_1}^{(1)}$. From equation (5.28) we have

$$(p_{\omega_1}^{(2)}, p_{\omega_2}^{(2)}) = \Gamma(\omega_1)\delta(\omega_1-\omega_2) \tag{5.56}$$

and

$$(p_{\omega_1}^{(1)}, p_{\omega_2}^{(1)}) = (1-\Gamma(\omega_1))\delta(\omega_1-\omega_2). \tag{5.57}$$

Using equations (5.38) and (5.56) leads to

$$\Gamma(\omega_1)\delta(\omega_1-\omega_2) = \int d\omega'(\alpha^*_{\omega_1\omega'}\alpha_{\omega_2\omega'} - \beta^*_{\omega_1\omega'}\beta_{\omega_2\omega'}), \tag{5.58}$$

where now $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are the Bogoliubov coefficients in the expansion of $p_\omega^{(2)}$ in terms of $f_{\omega'}$ and $f^*_{\omega'}$. As in equation (5.36) $b_\omega^{(2)} = (p_\omega^{(2)}, \phi)$.

In what follows we will no longer use the superscript (2) with the understanding that now $b_\omega = b_\omega^{(2)}$. The b_ω contain the information about the particles created during the collapse of the body. If we try to calculate

$$\langle 0|b^\dagger_\omega b_\omega|0\rangle = \int d\omega' |b_{\omega\omega'}|^2 \tag{5.59}$$

we find that the integral is infinite as a result of $\delta(\omega_1 - \omega_2)$ in equation (5.58). The term $\langle 0|b^\dagger_\omega b_\omega|0\rangle$ should be the total number of particles per unit angular frequency reaching $\mathcal{I}^+$. However, this flux of particles is steady in time and therefore some adjustment accounting for time must be made. To see this let

$$\delta(\omega_1 - \omega_2) = \lim_{t\to\infty} \frac{1}{2\pi} \int_{-t/2}^{t/2} dt' \; e^{i(\omega_1-\omega_2)t'}. \tag{5.60}$$

Then if $\omega_1 = \omega_2 = \omega$, and using equation (5.54)

$$\lim_{t\to\infty} \Gamma(\omega)\frac{t}{2\pi} = \int d\omega'(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2) = (e^{8\pi M\omega} - 1)\int d\omega' |\beta_{\omega\omega'}|^2. \tag{5.61}$$

We then obtain

$$\langle 0|b^\dagger_\omega b_\omega|0\rangle = \lim_{t\to\infty} \frac{t}{2\pi}\; \Gamma(\omega)(e^{8\pi M\omega} - 1)^{-1}. \tag{5.62}$$

We can view this result as the number of particles created per unit angular frequency per unit time which, at late times, pass through a surface of radius much greater than that of the event horizon. This flux of created particles is equal to

$$\frac{1}{2\pi}\Gamma(\omega)(e^{8\pi M\omega} - 1)^{-1}. \tag{5.63}$$

In equation (5.63) $\Gamma(\omega)$ is the fraction of an outgoing wave packet that if extended backward in time would pass through the collapsing body just before the formation of the event horizon. However, if the collapsing body spacetime were replaced with the analytic extension of the black hole spacetime, then this fraction is the same fraction of the wave packet that would enter the black hole past event horizon at late times (Hawking, 1975). This means that $\Gamma(\omega)$ is also the fraction of an incoming wave packet from $\mathcal{I}^-$ that would be absorbed by the black hole. The implication, then, is that a Schwarzschild black hole absorbs and emits radiation exactly like a gray body of absorptivity $\Gamma(\omega)$ and with temperature, T, given by

$$k_B T = (8\pi M)^{-1} = \frac{\kappa}{2\pi}, \tag{5.64}$$

where k_B is Boltzmann's constant and $\kappa = \frac{1}{4M}$ is the surface gravity of the black hole.

5.2 Hawking Radiation for Massless Fields for Higher-Derivative Klein-Gordon Equation

The Klein-Gordon equation in spherical coordinates for massless fields is

$$(-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\phi] = 0, \tag{5.65}$$

where $g^{\mu\nu}$ is the metric tensor, $(-g)$ is the determinant of $g_{\mu\nu}$ and ϕ is the solution to the equation (Parker and Toms, 2009).

In the case of a Schwarzschild black hole the metric exterior to the event horizon is

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{5.66}$$

Near the event horizon the solution to the K-G equation is not expressible with elementary functions, but at great distance and time the Schwarzschild metric becomes asymptotically Minkowski.

The field solution of the K-G equation can be written for the entire spacetime outside the event horizon as

$$\phi = \int d\omega(a_\omega f_\omega + a_\omega^\dagger f_\omega^*), \tag{5.67}$$

where the f_ω and f_ω^* are a complete set of solutions to the K-G equation and the a_ω and $a_\omega^\dagger$ are time-independent operators. Further, the inner product is defined as

$$(f_{\omega_1}, f_{\omega_2}) \equiv i \int d^3x |g|^{1/2} g^{0\nu} f^*_{\omega_1}(x,t) \overleftrightarrow{\partial}_\nu f_{\omega_2}(x,t), \tag{5.68}$$

and f_{ω_1} and f_{ω_2} satisfy

$$(f_{\omega_1}, f_{\omega_2}) = \delta(\omega_1 - \omega_2). \tag{5.69}$$

The canonical commutation relations

$$[a_{\omega_1}, a^\dagger_{\omega_2}] = \delta(\omega_1 - \omega_2) \tag{5.70}$$

$$[a_{\omega_1}, a_{\omega_2}] = 0 = [a^\dagger_{\omega_1}, a^\dagger_{\omega_2}] \tag{5.71}$$

imply that a_ω are annihilation operators and $a^\dagger_\omega$ are creation operators.

In Minkowski space the K-G equation for a massless field using spherical coordinates is

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = \left(\frac{\partial^2}{\partial t^2} - \nabla_r^2 - \nabla_\Omega^2\right)\phi = 0, \tag{5.72}$$

where the Laplacian is split into its radial and angular parts. The general solution to this equation is

$$\phi = R(r) Y_l^{m_l}(\theta, \phi) e^{-i\omega t}, \tag{5.73}$$

where the Y's are spherical harmonics (and the ϕ in Y's argument is obviously not the same ϕ as the solution).

So we have

$$\left(\frac{\partial^2}{\partial t^2} - \nabla_r^2 - \nabla_\Omega^2\right) R(r) Y_l^{m_l}(\theta, \phi) e^{-i\omega t} = 0 \tag{5.74}$$

or

$$Y_l^{m_l}(\theta, \phi) e^{-i\omega t}\left(\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} - \frac{l(l+1)}{r^2} R + \omega^2 R\right) = 0. \tag{5.75}$$

($\omega^2 = \omega_0^2 + k^2$ but $\omega_0 = 0$ in massless case).

Therefore, the radial K-G equation is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \omega^2 \right) R = 0. \tag{5.76}$$

The solution is $R(r) = \sqrt{\frac{2}{\pi}}\, \omega j_l(\omega r)$, where $j_l(\omega r)$ is the spherical Bessel function.

Again using spherical coordinates, the higher-derivative K-G equation is

$$\left(\frac{\partial^2}{\partial t^2} - \nabla_r^2 - \nabla_\Omega^2 \right)^2 \phi = 0, \tag{5.77}$$

where now ϕ is a solution to a fourth order differential equation. This equation has mixed derivatives of the time function, the radial function and the angular function. However, when considering Hawking radiation we are observing at late times, i.e. at great distances and times from the black hole. Thus we are observing basically at one spatial point at a large radial distance from the black hole. As Parker and Toms (2009) do, we can consider the radiation in the equatorial plane of the black hole ($\theta = \frac{\pi}{2}$) and at a fixed angular coordinate ϕ. This has the advantage of eliminating derivatives with respect to θ and to ϕ as well as eliminating the separation constants m_l and $l(l+1)$. The equation we arrive at is

$$\left(\frac{d^4}{dr^4} + \frac{4}{r}\frac{d^3}{dr^3} + 2\omega^2 \frac{d^2}{dr^2} + \frac{4\omega^2}{r}\frac{d}{dr} + \omega^4 \right) R = 0. \tag{5.78}$$

Solutions to this are

$$R(r) = c_1 \frac{e^{-i\omega r}}{r} + c_2 \frac{e^{i\omega r}}{r} + c_3 e^{-i\omega r} + c_4 e^{i\omega r}, \tag{5.79}$$

where the c's are constants.

The first and second terms are solutions to the standard K-G equation. The third and fourth terms are additional solutions due to the higher-derivative K-G. So in the H-D K-G equation the field solutions will have one part, ϕ_1, from the solutions to the standard K-G equation and another part, ϕ_2, from the additional solutions to the H-D K-G equation, i.e. $\phi = \phi_1 + \phi_2$. The ϕ_1 solutions have already been dealt with in Parker and Toms (2009). In what follows for the rest of this section, the operators, functions, constants, etc. are understood to be applying to the ϕ_2 solution. Thus, ϕ_2 can be written

$$\phi_2 = \int d\omega (a_\omega f_\omega + a_\omega^\dagger f_\omega^*). \tag{5.80}$$

On $\mathcal{I}^+$ p_ω and p_ω^* form a complete set of solutions

$$p_\omega = \int d\omega' (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f_{\omega'}^*), \tag{5.81}$$

where $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are Bogoliubov coefficients.

Asymptotic forms of f_ω and p_ω are

$$f_\omega \sim \omega^{-\frac{1}{2}} e^{-i\omega v} Y(\theta, \phi), \tag{5.82}$$

$$p_\omega \sim \omega^{-\frac{1}{2}} e^{-i\omega u} Y(\theta, \phi), \tag{5.83}$$

where $v = t + r$ and $u = t - r$.

The Bogoliubov coefficients are related to f_ω and p_ω by

$$\alpha_{\omega\omega'} = (f_{\omega'}, p_\omega) = C \int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{i\omega' v} e^{-i\omega u} \tag{5.84}$$

$$\beta_{\omega\omega'} = -(f_{\omega'}^*, p_\omega) = C \int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{-i\omega' v} e^{-i\omega u}, \tag{5.85}$$

where C is a constant.

Following Parker and Toms (2009), for null geodesic coordinates u and v with an affine parameter, u can be represented as a function of v, i.e.

$$u(v) = -4M \ln \left(\frac{v_0 - v}{K}\right), \tag{5.86}$$

where K is a positive constant. Now substitute for $u(v)$ and let $s \equiv v_0 - v$ in equation (5.84) and $s \equiv v - v_0$ in equation (5.85). Then from equation (5.84)

$$\alpha_{\omega\omega'} = C \int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{i\omega' v} e^{i\omega 4M \ln(\frac{v_0 - v}{K})}$$
$$= -C \int_{\infty}^{0} ds \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{-i\omega' s} e^{i\omega' v_0} e^{i\omega 4M \ln(\frac{s}{K})} \quad (5.87)$$

and from eq.(5.85)

$$\beta_{\omega\omega'} = C \int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{-i\omega' v} e^{i\omega 4M \ln(\frac{v_0 - v}{K})}$$
$$= C \int_{-\infty}^{0} ds \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{-i\omega' s} e^{-i\omega' v_0} e^{i\omega 4M \ln(-\frac{s}{K})}. \quad (5.88)$$

That the integrals in equations (5.87) and (5.88) converge is shown in an addendum. Continuing to follow Parker and Toms (2009) $\alpha_{\omega\omega'}$ can be integrated along the positive s-axis, a quarter circle at infinity around the 4th quadrant and then along the imaginary axis from $-i\infty$ to 0. The $\beta_{\omega\omega'}$ integral's contour will be along the real axis, a quarter circle at infinity around the 3rd quadrant and then along the imaginary axis from $-i\infty$ to 0. In both cases the integral along the imaginary axis is equal to that along the real axis, so we substitute $s \equiv is'$. Therefore, we have

$$\alpha_{\omega\omega'} = -iC \int_{-\infty}^{0} ds' \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{\omega' s'} e^{i\omega' v_0} e^{i\omega 4M \ln(\frac{is'}{K})} \quad (5.89)$$

$$\beta_{\omega\omega'} = iC \int_{-\infty}^{0} ds' \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{\omega' s'} e^{-i\omega' v_0} e^{i\omega 4M \ln(\frac{-is'}{K})}. \quad (5.90)$$

Taking a branch cut along the negative real axis to make the logarithm single-valued and noting that $s' < 0$ we have

$$\ln\left(\frac{is'}{K}\right) = \ln\left(\frac{-i|s'|}{K}\right) = -i\frac{\pi}{2} + \ln\left(\frac{|s'|}{K}\right) \quad (5.91)$$

and

$$\ln\left(\frac{-is'}{K}\right) = \ln\left(\frac{i|s'|}{K}\right) = i\frac{\pi}{2} + \ln\left(\frac{|s'|}{K}\right). \quad (5.92)$$

Then

$$\alpha_{\omega\omega'} = -iCe^{i\omega' v_0}e^{2\pi\omega M}\int_{-\infty}^{0} ds'\left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{\omega' s'}e^{i\omega 4M\ln(\frac{|s'|}{K})}, \tag{5.93}$$

and

$$\beta_{\omega\omega'} = iCe^{-i\omega' v_0}e^{-2\pi\omega M}\int_{-\infty}^{0} ds'\left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 e^{\omega' s'}e^{i\omega 4M\ln(\frac{|s'|}{K})}. \tag{5.94}$$

Thus, after squaring the absolute value of $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ and dividing α by β we find that

$$|\alpha_{\omega\omega'}|^2 = e^{8\pi M\omega}|\beta_{\omega\omega'}|^2. \tag{5.95}$$

This result is for the wave packet traced back in time through the collapsing body just before the event horizon formed.

The wave packet, p_ω, that reaches $\mathcal{I}^+$ is formed in two parts. Tracing p_ω backward in time, one part, $p_\omega^{(1)}$, is scattered by the spacetime around the collapsing body and reaches $\mathcal{I}^-$ with the same frequency, ω, that it had on $\mathcal{I}^+$. The other part, $p_\omega^{(2)}$, passes through the collapsing body and reaches $\mathcal{I}^-$. The two parts are disjoint regions on $\mathcal{I}^-$, $v > v_0$ for $p_\omega^{(1)}$ and $v < v_0$ for $p_\omega^{(2)}$. So $p_\omega = p_\omega^{(1)} + p_\omega^{(2)}$ and

$$(p_{\omega_1}, p_{\omega_2}) = (p_{\omega_1}^{(1)}, p_{\omega_2}^{(1)}) + (p_{\omega_1}^{(2)}, p_{\omega_2}^{(2)}). \tag{5.96}$$

If $\Gamma(\omega_1)$ is the fraction of p_{ω_1} corresponding to $p_{\omega_1}^{(2)}$, then $1 - \Gamma(\omega_1)$ is the fraction corresponding to $p_{\omega_1}^{(1)}$. From equation (5.28) we have

$$(p_{\omega_1}^{(2)}, p_{\omega_2}^{(2)}) = \Gamma(\omega_1)\delta(\omega_1 - \omega_2) \tag{5.97}$$

and

$$(p_{\omega_1}^{(1)}, p_{\omega_2}^{(1)}) = (1 - \Gamma(\omega_1))\delta(\omega_1 - \omega_2). \tag{5.98}$$

Using equations (5.38) and (5.56) leads to

$$\Gamma(\omega_1)\delta(\omega_1 - \omega_2) = \int d\omega'(\alpha^*_{\omega_1\omega'}\alpha_{\omega_2\omega'} - \beta^*_{\omega_1\omega'}\beta_{\omega_2\omega'}), \tag{5.99}$$

where now $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are the Bogoliubov coefficients in the expansion of $p_\omega^{(2)}$ in terms of $f_{\omega'}$ and $f^*_{\omega'}$. As in equation (5.36) $b_\omega^{(2)} = (p_\omega^{(2)}, \phi)$.

In what follows we will no longer use the superscript (2) with the understanding that now $b_\omega = b_\omega^{(2)}$. The b_ω contain the information about the particles created during the collapse of the body. If we try to calculate

$$\langle 0|b^\dagger_\omega b_\omega|0\rangle = \int d\omega' |b_{\omega\omega'}|^2 \tag{5.100}$$

we find that the integral is infinite as a result of $\delta(\omega_1 - \omega_2)$ in equation (5.99). The term $\langle 0|b^\dagger_\omega b_\omega|0\rangle$ should be the total number of particles per unit angular frequency reaching $\mathcal{I}^+$. However, this flux of particles is steady in time and therefore some adjustment accounting for time must be made. To see this let

$$\delta(\omega_1 - \omega_2) = \lim_{t\to\infty} \frac{1}{2\pi} \int_{-t/2}^{t/2} dt' \ e^{i(\omega_1-\omega_2)t'}, \tag{5.101}$$

where t and t' are time. Then if $\omega_1 = \omega_2 = \omega$, and using equation (5.95)

$$\lim_{t\to\infty} \Gamma(\omega)\frac{t}{2\pi} = \int d\omega' (|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2) = (e^{8\pi M\omega} - 1) \int d\omega' |\beta_{\omega\omega'}|^2. \tag{5.102}$$

We then obtain

$$\langle 0|b^\dagger_\omega b_\omega|0\rangle = \lim_{t\to\infty} \frac{t}{2\pi} \ \Gamma(\omega)(e^{8\pi M\omega} - 1)^{-1}. \tag{5.103}$$

We can view this result as the number of particles created per unit angular frequency per unit time which, at late times, pass through a surface of radius much greater than that of the event horizon. This flux of created particles is equal to

$$\frac{1}{2\pi}\Gamma(\omega)(e^{8\pi M\omega} - 1)^{-1}. \tag{5.104}$$

In equation (5.104) $\Gamma(\omega)$ is the fraction of an outgoing wave packet that if extended backward in time would pass through the collapsing body just before the formation of the event horizon. However, if the collapsing body spacetime were replaced with the analytic extension of the black hole spacetime, then this fraction is the same fraction of the wave packet that would enter the black hole past event horizon at late times. This means that $\Gamma(\omega)$ is also the fraction of an incoming wave packet from $\mathcal{I}^-$ that would be absorbed by the black hole. The implication, then, is that a Schwarzschild black hole absorbs and emits radiation exactly like a gray body of absorptivity $\Gamma(\omega)$ and with temperature, T, given by

$$k_B T = (8\pi M)^{-1} = \frac{1}{2\pi}\kappa, \tag{5.105}$$

where k_B is Boltzmann's constant and $\kappa = \frac{1}{4M}$ is the surface gravity of the black hole. This additional radiation due to the second part of the higher-derivative Klein-Gordon equation is equal to the radiation from the ordinary Klein-Gordon equation. Thus, it appears that the higher-derivative Klein-Gordon equation simply doubles the black hole Hawking radiation due to the ordinary Klein-Gordon equation for massless spin-0 particles.

It appears that increasing higher order application of the Klein-Gordon equation to the wave function leads to an increasingly unlimited emission of massless spin-0 particles from the black hole. This, of course, is physically impossible, but it is also a moot point as there are no known massless spin-0 particles.

5.3 Hawking Radiation from Schwarzschild Black Holes for Massive Field

To see what Hawking radiation is for massive particles in curved spacetime we must consider QFT in curved spacetime in more detail. Historically in QFT, positive frequency modes, f, are defined as satisfying $\frac{\partial f}{\partial t} = -i\omega f$, where $\omega > 0$. In Minkowski spacetime there is no ambiguity about the definition of positive frequency modes which provide a complete basis for the solutions to the equation of motion. In curved spacetime the Klein-Gordon equation given by many in the field (e.g. Hawking, 1975; Parker and Toms, 2009; Birrell and Davies, 1982) is

$$(-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\phi] + m^2\phi = 0 \tag{5.106}$$

and a positive frequency mode definition will be arbitrary so it will not be possible to define unambiguously a vacuum nor a number operator. The question then becomes, what would a particle detector detect traveling along some trajectory in curved spacetime. It measures the proper time, τ, and "defines" positive frequencies with respect to that proper time. If a set of modes can be found such that

$$\frac{D}{d\tau} f_i = -i\ \omega f_i \tag{5.107}$$

we could use these as the positive frequency modes. In general these will not exist for all spacetime. But in the case of a static spacetime we will have a hypersurface-orthogonal (timelike) Killing vector and there will exist a metric whose components are independent of the time coordinate, t. In that case positive frequency modes can be well defined and there are solutions which can be separated into time-independent and space-independent factors, thus allowing unambiguous definitions of number operators and vacuums. The Schwarzschild metric is such a static spacetime metric; its coordinate components are time independent and there are no space-time cross terms, i.e.

$$\partial_0 g_{\mu\nu} = 0, \qquad g_{0i} = 0, \tag{5.108}$$

where i is a spatial component index.

As pointed out on page 36, just before equation (5.14), the spectrum of the created particles of Hawking radiation are determined by the form of $u(\lambda)$. It is for this reason that for the null coordinate, u, of equation (5.18) we use an approximation (equation (5.19)) made near the event horizon. That is,

$$u \approx -4M \ln\left(\frac{\lambda}{K_1}\right) = -4M \ln\left(\frac{v_0 - v}{K_1 K_2}\right), \tag{5.109}$$

where $K_1 K_2 > 0$, near the event horizon.

For the Schwarzschild metric, there are two constants of the motion

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = E \quad \text{and} \quad r^2 \frac{d\phi}{d\lambda} = L, \tag{5.110}$$

where λ is an affine parameter.

Using the constants of the motion, the geodesic equation for a massless particle traveling radially ($L = 0$) in this metric is

$$\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 = 0 \tag{5.111}$$

or,

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 \tag{5.112}$$

and therefore

$$\frac{dr}{d\lambda} = \pm E. \tag{5.113}$$

Equation (5.113) then defines null geodesics in the Schwarzschild metric, along which massless particles travel with speed c, the speed of light. However, massive particles travel on timelike geodesics and the geodesic equation in this case will be

$$\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = 1, \tag{5.114}$$

where τ is the proper time. Again, using the constants of the motion with $L = 0$, we have

$$E^2 - \left(\frac{dr}{d\tau}\right)^2 = \left(1 - \frac{2M}{r}\right) \tag{5.115}$$

or, after rearranging

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - 1 + \frac{2M}{r} \tag{5.116}$$

which is clearly different from equation (5.112) by the addition of $-1 + \frac{2M}{r}$ on the RHS.

As noted above the creation of particles by the intense gravitational field of the black hole will occur just outside the event horizon and the minimum velocity of these particles will be the escape velocity (otherwise they will not escape to infinity). Of course, the velocity could be greater than the escape velocity, but we will use $v = v_{esc}$ in the following calculations as that will be the minimum energy condition to determine the particle geodesic equation. The escape velocity is

$$v_{esc} = \left(\frac{2M}{r}\right)^{\frac{1}{2}}, \tag{5.117}$$

where $r \geq 2M$ and at the event horizon $v_{esc} = 1$, the speed of light. The massive particles created will have velocities close to the speed of light and their timelike geodesics will be very close to null geodesics. To see this let $r = 2M + \epsilon$, where ϵ is very small and $2M \gg \epsilon > 0$. Then the timelike geodesic equation will be

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - 1 + \frac{2M}{r} = E^2 - 1 + \frac{2M}{2M+\epsilon} = E^2 - \frac{\epsilon}{2M+\epsilon} \cong E^2 - \frac{\epsilon}{2M}. \tag{5.118}$$

To show that this is close to a null geodesic we need to show that $\dfrac{\epsilon}{2M} \ll E^2$ and so we need an estimate of the size of E^2. We note that

$$E^2 = k^2 + m^2 = m^2\gamma^2 v^2 + m^2 = m^2(\gamma^2 v^2 + 1), \tag{5.119}$$

where k is the particle momentum, m is the rest mass of the particle and γ^2 is

$$\gamma^2 = \frac{1}{1 - v^2} = \frac{1}{1 - \frac{2M}{2M+\epsilon}} = \frac{2M+\epsilon}{\epsilon} = \frac{2M}{\epsilon} + 1. \tag{5.120}$$

So

$$E^2 = m^2(\gamma^2 v^2 + 1) = m^2 \left[\left(\frac{2M+\epsilon}{\epsilon} \right) \left(\frac{2M}{2M+\epsilon} \right) + 1 \right]$$

$$= m^2 \left(\frac{2M}{\epsilon} + 1 \right) \cong \frac{2Mm^2}{\epsilon}, \quad \text{since } \frac{2M}{\epsilon} \gg 1. \tag{5.121}$$

Finally, from equation (5.118) (for $r = 2M + \epsilon$)

$$\left(\frac{dr}{d\tau} \right)^2 \cong E^2 - \frac{\epsilon}{2M} \cong \frac{2Mm^2}{\epsilon} - \frac{\epsilon}{2M}. \tag{5.122}$$

For equation (5.122) to be close to E^2 (null geodesic) we need

$$\frac{\epsilon}{2M} \ll \frac{2Mm^2}{\epsilon}$$

$$\epsilon^2 \ll 4M^2 m^2$$

$$\epsilon \ll 2Mm. \tag{5.123}$$

For ϵ to be much less than $2Mm$ there must exist a large positive number, N, such that $\epsilon = \dfrac{2Mm}{N}$.

Then from equation (5.121) $E^2 \cong \dfrac{2Mm^2}{\epsilon} = \dfrac{2Mm^2}{\frac{2Mm}{N}} = Nm$

and from equation (5.118) $-1 + \dfrac{2M}{r} \cong \dfrac{\epsilon}{2M} = \dfrac{\frac{2Mm}{N}}{2M} = \dfrac{m}{N}$ (for $r = 2M + \epsilon$). Since N is a large positive number then $Nm \gg \dfrac{m}{N}$ and $E^2 \gg \dfrac{\epsilon}{2M}$. Thus,

$$\left(\frac{dr}{d\tau} \right)^2 \cong E^2 - \frac{\epsilon}{2M} \cong E^2 \tag{5.124}$$

and the geodesic equation for massive particles created near the event horizon is very close to the null geodesic. As the created particles distance themselves from the black hole and their velocities decrease their geodesics will deviate more and more from a null geodesic, but the particles, by then, are already created. It is during this early phase of particle creation that determines the spectrum of created particles observed at late times. The massive particles created will have a dispersion relation $\omega = \sqrt{k^2 + m^2}$. The massless particles have a dispersion relation $\omega = k$. The k's in both dispersion relations will have the same spectrum of values. Thus, the spectrum of massive particles created will be the same as that of massless particles created with the only difference being the difference in the dispersion relation between the two. The net result is that in the expression for the flux of created particles observed at late times is just equation (5.63), i.e.

$$\frac{1}{2\pi}\Gamma(\omega)(e^{8\pi M\omega} - 1)^{-1}, \tag{5.125}$$

but with the ω for massless particles, $\omega = k$, now replaced with the ω for massive particles, $\omega = \sqrt{k^2 + m^2}$. The spectrum of particles will still be a thermal spectrum given by equation (5.125).

As an example of what particles might be created and their likelihood of creation we note that we are dealing with the Klein-Gordon equation whose solutions are scalar functions and the fields are spin-0. For an intermediate mass black hole on the order of a solar mass, the event horizon temperature is $\sim 10^{-8}$ K. As Hawking (1975) pointed out this is much less than the cosmic microwave background (CMB) temperature and creation of any massive particle would be extremely rare. But for much smaller black holes, perhaps primordial black holes, the event horizon surface temperature is much higher, permitting a greater probability of massive particle creation. The only known spin-0 particle is the Higgs boson. If we consider a solar mass black hole, then using the Stefan-Boltzmann law of blackbody radiation, the event horizon surface temperature in ordinary units is

$$T = \frac{\hbar\ c^3}{8\ \pi\ G\ k_B\ M_\odot} \cong 6 \times 10^{-8}\ \text{K}, \tag{5.126}$$

where $\hbar$ is the reduced Planck constant, c is the speed of light, G is the gravitational constant, k_B is Boltzmann's constant and $M_\odot$ is the solar mass. This represents an energy of

$$k_B T \cong 8 \times 10^{-31} \text{Joules}, \tag{5.127}$$

whereas for the Higgs particle the mass energy is $\sim$ 125 GeV $\cong 2 \times 10^{-8}$ Joules, exceeding the energy at the event horizon of a solar mass black hole by roughly 23 orders of magnitude. Thus, the creation of a Higgs particle near the event horizon of a solar mass black hole is essentially zero. However, if the black hole is very small, say on the order of 10^8 kg, then the event horizon temperature is orders of magnitude higher so that the energy at the event horizon is comparable to the Higgs mass energy and there is the likelihood of the creation of Higgs particles.

Zel'dovitch and Novikov (1966) first proposed that in the early universe, shortly after the big bang, there were numerous black holes, called primordial black holes (PBH). Later, Hawking (1971) and others made similar predictions. These PBH would have long ago radiated away their masses, the final fraction of a second producing an explosion releasing enormous amounts of energy. Some of the particles created during this explosion could have been Higgs particles. If there had been large numbers of these PBH, their disappearance due to radiation could possibly account for the appearance of Higgs particles.

5.4 **Various Possibilities of Tachyonic Creation by Black Holes**

As pointed out in chapter 3, the even order series higher-derivative Klein-Gordon equation yields the Klein-Gordon equation for ordinary matter as well as the "negative" Klein-Gordon equation ($-m^2$ instead of $+m^2$) without appeal to the Special Theory of Relativity or any assumptions about the velocity of reference frames. We can interpret the solutions to that equation ($-m^2$) as faster-than-light particles (tachyons). Among the first to consider the idea of particles which travel faster than light was Sommerfeld who rejected the notion of the existence of these particles. In 1967 Gerald Feinberg (1967) studied the possibility of faster than light particles and coined the term "tachyon." Since then many physicists have devoted time to theorizing about tachyons and trying (without success) to detect them. The following sections (5.4.1, 5.4.2, 5.4.3 and 5.4.4) present various views of tachyons and tachyon creation by black holes, but they are not meant to be an exhaustive listing of such possibilities.

5.4.1 Description of Tachyons

Bilaniuk and Sudarshan (1969) placed all particles into three classes. Class I are slower than light particles called tardyons, bradyons or ordinary matter. Class II are particles which travel at the speed of light (and, of course, only at the speed of light) called luxons. Class III are the faster than light particles (tachyons).

A class I particle obeys the relation $E^2 - p^2c^2 = m_0^2c^4$, where E is the energy, p is the momentum, m_0 is the proper ("rest") mass and c is the speed of light. For simplicity and without loss of generality, let's assume that this particle moves along the x-axis, so $p = p_x$ and $p_y = p_z = 0$. Then

$$E - p_x^2c^2 = m_0^2c^4, \tag{5.128}$$

which describes a hyperbola in E and p_x coordinates (see figure 5.2). The positive energy (upper curve) and the negative energy (lower curve) states are separated and there is no transformation from the upper curve to the lower curve. In any case, negative energy class I particles would not be observable. For momentum $p_x = 0$ the energy will be $E = m_0c^2$, called the proper energy.

For class II particles (luxons) the proper mass is zero and so

$$E^2 - p_x^2 = (E + p_x)(E - p_x) = 0, \tag{5.129}$$

which represents the two intersecting straight lines in figure 5.2.

Now for class III particles the Special Theory of Relativity (SR) requires that

$$E = mc^2 \tag{5.130}$$

and

$$m = \frac{m_0}{\left[1 - (\frac{v}{c})^2\right]^{1/2}}, \tag{5.131}$$

where m is the relativistic mass. There is no concern about imaginary values for class I particles (all values are real), but for class III particles $\left[1 - (\frac{v}{c})^2\right]^{1/2}$ will be imaginary since $\frac{v}{c} > 1$. For this reason the proper mass is defined as imaginary, with

$$m_0 = i\ m_*, \tag{5.132}$$

where m_0 = proper mass and m_* is called the meta-mass. The relativistic mass in equation (5.131) now becomes

$$m = \frac{m_0}{\left[1 - (\frac{v}{c})^2\right]^{1/2}} = \frac{im_*}{\left[1 - (\frac{v}{c})^2\right]^{1/2}} = \frac{m_*}{\left[(\frac{v}{c})^2 - 1\right]^{1/2}}, \tag{5.133}$$

and m remains real. Figure 5.2 is a graph of energy, E, vs. momentum, p_x, for class I, II, III particles.

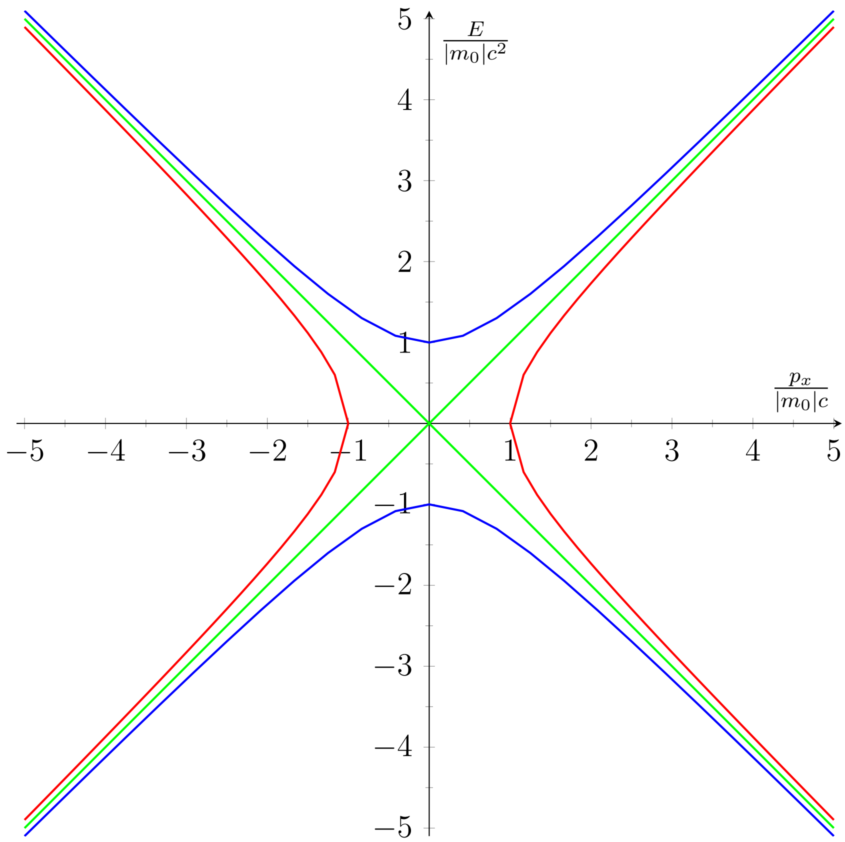

Figure 5.2 These graphs are relations between energy, E, and momentum, p_x; (blue for bradyons, red for tachyons and green for luxons)

As pointed out above for class I particles, positive energy and negative energy are separated and there is no transformation of one to the other. For tachyons, however, positive and negative energies are not separated, as seen by the red hyperbola in figure 5.2. As a further explanation, let S be a system in which we are at rest and S' a system with velocity w with respect to system S. Then a tachyon traveling with velocity v in the S frame will have a different velocity, call it u, with respect to frame S'. If the product vw becomes greater than c^2 the point on the hyperbola representing the state of the particle in S' will be in the negative energy realm and the particle will appear to have been absorbed first and emitted later. This can be seen in figure 5.3.

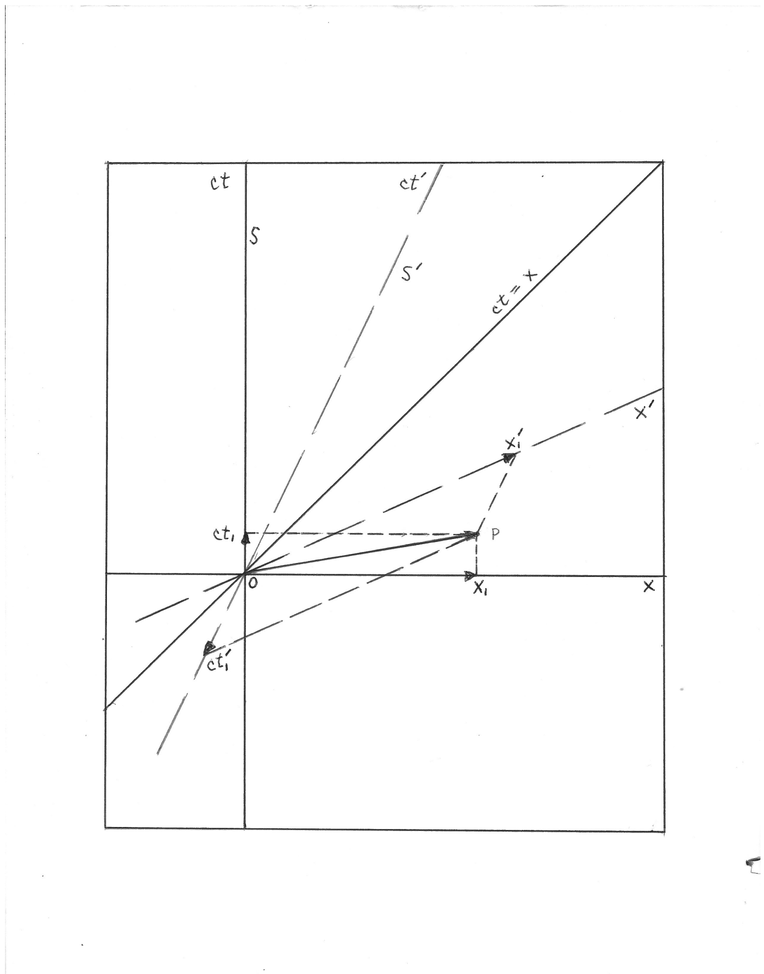

Figure 5.3 Space-time Diagram

This figure shows graphs of two sets of axes in S and S'. Point O is the emission of a tachyon and point P is the absorption of the tachyon. It can be seen from the figure that in the unprimed frame P occurs after O, but in the primed frame O occurs after P, i.e. the tachyon is absorbed before it is emitted. This sequence reversal will happen whenever point P is below the x'-axis, which is also when the energy is negative.

The two aspects of these events in S', i.e. negative energy and traveling backward in time, are resolved by what is termed the reinterpretation principle. This principle states that a particle with negative energy traveling backward in time is equivalent to a positive energy particle traveling forward in time.

5.4.2 Tachyons produced by Schwarzschild Black Holes

In the 1970's and early 1980's a number of researchers considered the emission of tachyons from black holes. In 1983 Srivastava (1983) noted that the metric inside the event horizon (EH) of a Schwarzschild black hole is spacelike and assumed that tachyons are in a superdense state inside the EH. The equation of state then is $p = \rho$, where p is the pressure and ρ is the energy density of a tachyonic perfect fluid. As proposed by Mignani and Recami (1976), for tachyons time is a vector quantity, but position becomes essentially a scalar. The line element in this case derived from Einstein's equations is

$$\begin{aligned} ds^2 &= \frac{(1+\tau)^2}{2r} dr^2 - \frac{2r}{(1+\tau)^2}(d\tau_x^2 + d\tau_y^2 + d\tau_z^2) \\ &= \frac{(1+ct)^2}{2r} dr^2 - \frac{2rc^2}{(1+ct)^2}(dt_x^2 + dt_y^2 + dt_z^2), \end{aligned} \tag{5.134}$$

where $\tau_i = ct_i$, $\tau^2 = \tau_x^2 + \tau_y^2 + \tau_z^2$ and $t^2 = t_x^2 + t_y^2 + t_z^2$.

For free spin-0 tachyons the equation to be satisfied would be the Klein-Gordon equation which is

$$\left\{ (-g)^{-1/2} \frac{\partial}{\partial x^\mu} \left[(-g)^{1/2} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right] + m^2 \right\} \psi(r, t_x, t_y, t_z) = 0, \tag{5.135}$$

where $g_{\mu\nu}$ is the metric given by equation (5.134), $x^\mu = (r, t_x, t_y, t_z)$ and $g = -\frac{4r^2c^6}{(1+ct)^4}$. Plane wave solutions to (5.135) will be of the form $\psi = \frac{f(r)}{r} e^{-kt}$, where $kt = k_1 t_x +$

$k_2 t_y + k_3 t_z$ and $k = (k_1, k_2, k_3)$ are constants. This will yield an ordinary differential equation

$$\frac{d^2 f}{dr^2} + \left[\frac{(1+ct)^4 k^2}{4c^2 r^2} + \frac{m^2(1+ct)^2}{2r}\right] f = 0 \tag{5.136}$$

for a given t. The solution is found using WKB approximation and the transmission and reflection coefficients are calculated. It turns out that they are equal and that half of the tachyons incident on the EH are transmitted across the EH and therefore tachyons are produced by the black hole. Srivastavas' calculations also show that as the tachyon recedes from the black hole its momentum and energy decrease and that there is a certain distance and time at which the energy of the tachyon will vanish.

5.4.3 Quantization of Black Holes

He and Ma (2011) develop a notion of black hole quantization by considering that black holes are characterized by only three properties: M, the mass of the black hole, Q, the electric charge of the black hole and J, the angular momentum of the black hole. Schwarzschild black holes have no electric charge and no angular momentum. So for such a black hole (BH) other quantities such as the radius, $R = \frac{2GM}{c^2}$, the surface area, $A = 4\pi R^2$ and the Compton wavelength, $\lambda = \frac{\hbar}{Mc}$ calculated from the properties of BH's can only depend on M and are therefore essentially the same quantity. The equality

$$R\lambda = 2l^2, \tag{5.137}$$

where $l = \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length, $\hbar$ is the reduced Planck constant, c is the speed of light and G is the universal gravitational constant, is true for any mass, M.

Since nothing inside the Schwarzschild radius, R, can be known to an outside observer, all the information of the BH should be considered as residing on the surface, the event horizon or holographic screen from the holographic principle as described by Susskind

(1994). BH's create and emit particles with a thermal spectrum and temperature $T = \frac{\hbar}{8\pi Mck}$, where k is the Boltzmann constant.

Using Bohr's theory of quantization of the hydrogen atom one can make a similar argument that the BH behaves like a wave with Compton wavelength $\lambda = \frac{\hbar}{Mc}$ and that its wave be a standing wave on a spherical surface so that

$$R = 2n\lambda. \tag{5.138}$$

Using equation (5.137) with equation (5.138) we get

$$R_n = 2\sqrt{n}\, l, \tag{5.139}$$

and we see that the radius is quantized. Other quantities are now seen to be quantized:

$$E_n = M_n c^2 = \frac{R_n c^4}{2G} = \sqrt{n} M_P c^2 \tag{5.140}$$

$$A_n = 4\pi R_n^2 \tag{5.141}$$

$$\lambda_n = \frac{\hbar}{M_n c} = \frac{l}{\sqrt{n}} \tag{5.142}$$

$$T_n = \frac{M_P c^2}{8\pi\sqrt{n} k}, \tag{5.143}$$

where $M_P = \sqrt{\frac{\hbar c}{G}}$ is the Planck mass.

The smallest stable BH would be for $n = 1$ and then $R_1 = 2l$. In other words the radius would be twice the Planck length, the energy, E_1, is just the Planck mass energy, $M_P c^2$, and the Compton wavelength is equal to the Planck length, $\lambda = l$. So the smallest BH is of the Planck scale, but n can be very large and thus BH's can be enormously large. As n gets larger the energy difference between successive states becomes smaller, i.e.

$$\Delta E = E_{n+1} - E_n = (\sqrt{n+1} - \sqrt{n}) M_P c^2 = \frac{M_P c^2}{\sqrt{n+1} + \sqrt{n}}. \tag{5.144}$$

This quantization of energy states for BH's would seem to suggest that BH's can emit as well as absorb particles.

5.4.4 Creation and Annihilation of Tachyons by Black Holes

Using the results of He and Ma (2011), Sahoo and Kumar (2012) claim that the quantization of energy states of BH's can be related to the relativistic mass of the combined masses of ordinary matter and tachyons. The relativistic mass for ordinary matter is

$$m_1 = \frac{m_{01}}{\sqrt{1-(\frac{v}{c})^2}}, \tag{5.145}$$

and for tachyons it is

$$m_2 = \frac{m_{02}}{\sqrt{(\frac{v}{c})^2-1}}, \tag{5.146}$$

where $m_{01},\ im_{02}$ are the respective rest masses and v is the particle velocity.

The total combined mass of the ordinary matter particle and the tachyon is

$$M = m_1 + m_2 = \frac{m_{01}}{\sqrt{1-(\frac{v}{c})^2}} + \frac{m_{02}}{\sqrt{(\frac{v}{c})^2-1}}. \tag{5.147}$$

Two different cases for the combined mass are defined.

Case I: $v < c$

$$M_P = \frac{m_{01}}{\sqrt{1-(\frac{v}{c})^2}} + \frac{m_{02}}{i\sqrt{(1-\frac{v}{c})^2}} = \frac{M_0}{\sqrt{1-(\frac{v}{c})^2}}\ e^{-i\phi}, \tag{5.148}$$

where $\phi = \tan^{-1}(\frac{m_{02}}{m_{01}})$ and $M_0 = \sqrt{m_{01}^2 + m_{02}^2}$.

Case II: $v > c$

$$M_T = \frac{m_{01}}{i\sqrt{1-(\frac{v}{c})^2}} + \frac{m_{02}}{\sqrt{(1-\frac{v}{c})^2}} = \frac{-iM_0}{\sqrt{(\frac{v}{c})^2-1}}\ e^{i\phi}. \tag{5.149}$$

M_P, M_T are complex masses, whereas m_{01}, m_{02} are real masses.

Sahoo and Kumar (2012) show that BH's are quantized in discrete energy states (He and Ma, 2011) and that the combined masses can be correlated with different energy states. These different energy states determine the creation and annihilation of ordinary matter and tachyons. For ordinary matter the energy state of the BH would be

$$E_P = M_P c^2 = \frac{M_0 c^2}{\sqrt{1 - (\frac{v}{c})^2}}\ e^{-i\phi}, \tag{5.150}$$

whereas for tachyons it would be

$$E_T = M_T c^2 = \frac{-iM_0 c^2}{\sqrt{(\frac{v}{c})^2 - 1}}\ e^{i\phi}. \tag{5.151}$$

So if a BH is in state E_P then particles with $v < c$ are created and annihilated. In state E_T, particles with $v > c$ are created and annihilated.

Black holes of mass M behave like waves of Compton wavelength $\lambda = \frac{\hbar}{Mc}$ [16]. Further, Hawking showed that BH's create and emit particles as though the BH had a temperature $T \cong \frac{1}{8\pi M}$ [14]. The putative mechanism of this creation is that just outside the event horizon two particles are created, one with positive energy which escapes to infinity and the other with negative energy which enters the black hole.

Let's consider three special cases: $\phi = 0,\ \frac{\pi}{4},\ \frac{\pi}{2}$.

Case I: $\phi = 0$

$$E_P = \frac{M_0 c^2}{\sqrt{1 - (\frac{v}{c})^2}} \tag{5.152}$$

$$E_T = \frac{-iM_0 c^2}{\sqrt{(\frac{v}{c})^2 - 1}}. \tag{5.153}$$

In this case ordinary particle are created and tachyons are annihilated.

Case II: $\phi = \frac{\pi}{4}$

$$E_P = \frac{M_0 c^2}{\sqrt{1 - (\frac{v}{c})^2}}\, e^{\frac{-i\pi}{4}} = \frac{M_0 c^2}{\sqrt{1 - (\frac{v}{c})^2}} \left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] \tag{5.154}$$

$$E_T = \frac{-iM_0 c^2}{\sqrt{(\frac{v}{c})^2 - 1}}\, e^{\frac{-i\pi}{4}} = \frac{M_0 c^2}{\sqrt{(\frac{v}{c})^2 - 1}} \left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]. \tag{5.155}$$

The same number of ordinary matter particles are created as annihilated. Equally for tachyons, the same number are created as annihilated.

Case III: $\phi = \frac{\pi}{2}$

$$E_P = \frac{M_0 c^2}{\sqrt{1 - (\frac{v}{c})^2}}\, e^{\frac{-i\pi}{2}} = \frac{-iM_0 c^2}{\sqrt{1 - (\frac{v}{c})^2}} \tag{5.156}$$

$$E_T = \frac{-iM_0 c^2}{\sqrt{(\frac{v}{c})^2 - 1}}\, e^{\frac{-i\pi}{2}} = \frac{M_0 c^2}{\sqrt{(\frac{v}{c})^2 - 1}}. \tag{5.157}$$

In this case ordinary matter particles are annihilated, but tachyons are created.

The conclusion is that ordinary matter and tachyons can be created and annihilated by a BH depending on its (quantized) energy state (Sahoo and Kumar, 2012).

5.5 New Superluminal Special Relativity Transformations

Hill and Cox (2012) have proposed transformations (in fact, two new transformations) applied to inertial reference frames with relative velocities greater than the speed of light ($c < v < \infty$) which are, as they say, "complementary" to the Lorentz transformations of Special Relativity for subluminal velocities ($0 \leq v < c$). Further, with these transformations there is no need for contrived concepts such as imaginary mass nor for complicated physics. This extension of Special Relativity allows for faster than light motion.

We consider a rest frame with coordinates X, Y, Z, T and another frame with coordinates x, y, z, t moving with velocity v relative to the rest frame. We also use the standard convention of assuming that axis X and axis x are aligned with each other, that $Y = y$, $Z = z$, and that v is along the direction of the $X - x$ axis. Then the only coordinates of concern are x, t and X, T.

5.5.1 Subluminal Special Relativity

In Special Relativity the Lorentz transformations relating the rest frame, X, to the moving frame, x, are

$$X = \frac{x + vt}{\sqrt{1 - (v/c)^2}}, \qquad T = \frac{t + vx/c^2}{\sqrt{1 - (v/c)^2}} \tag{5.158}$$

and the inverse transformations are

$$x = \frac{X - vT}{\sqrt{1 - (v/c)^2}}, \qquad t = \frac{T - vX/c^2}{\sqrt{1 - (v/c)^2}}. \tag{5.159}$$

Of course, when $v = 0$, then

$$x = X, \qquad t = T, \qquad v = 0 \tag{5.160}$$

or

$$x + ct = X + cT, \qquad x - ct = X - cT, \qquad v = 0. \tag{5.161}$$

With $U = \dfrac{dX}{dT}$ and $u = \dfrac{dx}{dt}$ we have the well-known equation for addition of velocities

$$u = \frac{U - v}{1 - Uv/c^2}. \tag{5.162}$$

Besides equations (5.160) for $0 \leq v < c$ there is also the possibility of $x = -X, t = -T$ that would be consistent with $u = U$ and yielding the Lorentz transformations with X and T replaced with $-X$ and $-T$. However, such a replacement would mean a reversal of space and time which would be contrary to Galilean transformations for $v \ll c$. In the case for $c < v < \infty$ there is no such restriction on the sign changes and the Special Relativity formulation must depend on the mathematical structure.

5.5.2 Superluminal Special Relativity

Assuming that in the regime $c < v < \infty$ equation (5.162) still holds, then for $v \to \infty$ the relation $uU = c^2$ is true. This implies two possible constraints for the new transformations:

$$x = -cT, \quad t = -\frac{X}{c}, \quad v = \infty \tag{5.163}$$

or

$$x = cT, \quad t = \frac{X}{c}, \quad v = \infty. \tag{5.164}$$

The first constraint (equation (5.163)) will give the first possible new transformation

$$x = \frac{X - vT}{\sqrt{(v/c)^2 - 1}}, \quad t = \frac{T - vX/c^2}{\sqrt{(v/c)^2 - 1}}. \tag{5.165}$$

The second constraint (equation (5.164)) gives the second possible new transformation

$$x = \frac{-X + vT}{\sqrt{(v/c)^2 - 1)}}, \quad t = \frac{-T + vX/c^2}{\sqrt{(v/c)^2 - 1}}. \tag{5.166}$$

Hill and Cox (2012) show that the first transformation, equation (5.165), leads to energy-momentum relations which are invariant under this new transformation, i.e.

$$\text{Case I:} \quad m = \frac{p_\infty/c}{\sqrt{(v/c)^2 - 1}}, \qquad \mathcal{E} = mc^2 \tag{5.167}$$

In this case, p_∞ is the limiting value of $p = mv$ as $v \to \infty$. In other words, at $v = \infty$ the mass m will be zero, while the momentum p will have a finite, non-zero value. If we do not insist on energy-momentum invariance under transformation, then the second possible transformation, equation (5.169), will lead to the relations

$$\text{Case II:} \quad m = \frac{m_\infty v/c}{\sqrt{(v/c)^2 - 1}}, \qquad \mathcal{E} = \frac{1}{2}m(c^2 + v^2) - \frac{1}{2}m_\infty\, c^2\, \cosh^{-1}\left(\frac{v}{c}\right) + \mathcal{E}_0 \tag{5.168}$$

where m_∞ is the finite mass in the limit as v approaches infinity and $\mathcal{E}_0$ is an arbitrary constant.

In this dissertation we are considering only systems in which the Poincaré group applies which means that of the two cases mentioned above only the first case, for which the energy-momentum relations are invariant, is physically meaningful. Thus, in what follows, only the first case will be considered in detail.

5.6 Tachyonic Hawking Radiation

Particles of Hawking radiation are due to the quantum changes in the vacuum produced by the collapsing black hole. In what follows we will let $G = c = 1$ (geometrized units), so that GM/c^2 will simply be equal to M, or to put it differently,

M in the following equations would be GM/c^2 in SI units. The geodesic equation using the Schwarzschild metric is

$$\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2-\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2-r^2\left(\frac{d\theta}{d\lambda}\right)^2-r^2\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2=\pm1, \tag{5.169}$$

where $+1$ corresponds to ordinary matter and -1 corresponds to superluminal (tachyonic) matter. (For massless particles the RHS would be zero.) For both ordinary matter and tachyons we can use the proper time τ in place of the affine parameter λ that was used for massless particles on null geodesics. As pointed out in equation (5.110), there are two constants of motion, meaning that the Schwarzschild metric does not depend on two coordinates (t and ϕ). The motion of the created particle is in a plane (and is in fact radial), so we may orient the system so that $\theta=\pi/2$ and therefore $\sin\theta=1$ and $d\theta/d\tau=0$. The constants of the motion, then, are

$$\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)=E \tag{5.170}$$

$$r^2\left(\frac{d\phi}{d\tau}\right)=L, \tag{5.171}$$

where E and L are, respectively, the energy per unit mass and the angular momentum per unit mass. The geodesic equation can now be written

$$\left(1-\frac{2M}{r}\right)^{-1}E^2-\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2-\frac{L^2}{r^2}=\pm1 \tag{5.172}$$

or with a little algebra

$$E^2-\left(\frac{dr}{d\tau}\right)^2-\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}\pm1\right)=0. \tag{5.173}$$

After carrying out the implicit multiplications and solving for E^2

$$E^2=\left(\frac{dr}{d\tau}\right)^2+\left(\pm1\mp\frac{2M}{r}+\frac{L^2}{r^2}-\frac{2ML^2}{r^3}\right). \tag{5.174}$$

In our case there is no angular momentum, so $L = 0$ which leaves

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(\pm 1 \mp \frac{M}{r}\right) = \left(\frac{dr}{d\tau}\right)^2 \pm \left(1 - \frac{2M}{r}\right), \tag{5.175}$$

again, the + for ordinary matter and the − for tachyons. The Schwarzschild radius, r_s, is defined as $r_s = 2M$ and we rewrite equation (5.175) as

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 \pm \left(1 - \frac{r_s}{r}\right). \tag{5.176}$$

We can consider the views of two different observers: one located near the event horizon, the other at a great distance from the black hole. Both observers have the same metric and we write E^2 as

$$E^2 = v^2 \pm \left(1 - \frac{r_s}{r}\right), \tag{5.177}$$

where E is dimensionless, $(+)$ corresponds to ordinary matter ($v < 1$), $(-)$ corresponds to tachyonic matter ($v > 1$) and v is in units of c. Moreover, if e = energy $= mc^2 = m$ since we are using geometrized units with $G = c = 1$, then E must equal 1 since $E = e/m$ (energy per unit mass). So solving for v with $E = 1$ we have

$$v = \sqrt{1 \mp \left(1 - \frac{r_s}{r}\right)}, \tag{5.178}$$

which shows that in the limit as $r \to r_s$ (the Schwarzschild radius) equation (5.178) yields

$$v \;=\; 1 \quad \text{(in units of c)} \tag{5.179}$$

for ordinary particles as well as for tachyons.

To continue we revert back to SI units ($G \neq 1$, $c \neq 1$). If we assume that a local observer views the particles created at $r = r_s + \epsilon$, where ϵ is very small compared to r_s, then equation (5.177) becomes

$$E^2 = \left(\frac{v}{c}\right)^2 \pm \left(1 - \frac{r_s}{r_s + \epsilon}\right) = \left(\frac{v}{c}\right)^2 \pm \left(\frac{\epsilon}{r_s + \epsilon}\right) = 1, \tag{5.180}$$

$$\text{or} \qquad \left(\frac{v}{c}\right)^2 = 1 \mp \frac{\epsilon}{r_s + \epsilon}, \tag{5.181}$$

where now $(-)$ is for ordinary matter and $(+)$ is for tachyons. Finally, solving for v yields

$$v = \sqrt{1 \mp \frac{\epsilon}{r_s + \epsilon}} \;\; c. \tag{5.182}$$

We see, then, that for ordinary matter $v < c$ and for tachyons $v > c$ as would be expected and that the value of v depends on ϵ, the distance from the event horizon.

A distant observer cannot see the particle creation process since the time for this information to reach this observer approaches infinity. However, the distant observer sees a continuous flux of particles and is able to measure the speeds of the arriving particles. According to Hobson, Efstathiou, Lasenby (2006) the distant observer measures a characteristic energy of the particles generated by a black hole as

$$\mathcal{E} = k_b T = k_b \;\; \frac{\hbar \; c^3}{8\pi \; k_b \; G \; M} = \frac{\hbar \; c^3}{8\pi \; G \; M}. \tag{5.183}$$

From Hill and Cox (2012) $\mathcal{E} = mc^2$, so $\frac{\hbar \; c^3}{8\pi \; GM} = mc^2$ or, solving for m

$$m = \frac{\hbar \; c}{8\pi \; GM} = \left(\frac{\hbar}{4\pi \; c}\right) \left(\frac{c^2}{2GM}\right) = \frac{\hbar}{4\pi \; c \; r_s}. \tag{5.184}$$

This seems to say that the mass of the particles created by a black hole depends only on the inverse Schwarzschild radius and, hence inversely on the mass of the black hole $\left(r_s = \frac{2GM}{c^2}\right)$. Thus, the smaller the black hole (smaller M), the more massive the particles created.

Using the relativistic mass of particles of ordinary matter $\left(m = \frac{m_0}{\sqrt{1-(v/c)^2}}\right)$ and setting it equal to m in equation (5.184) allows us to calculate the velocities of particles of mass m. Then

$$\frac{m_0}{\sqrt{1 - (v/c)^2}} = \frac{\hbar}{4\pi \; c \; r_s} \tag{5.185}$$

or

$$\frac{4\pi\ r_s}{\lambda_c} = \sqrt{1-(v/c)^2}, \tag{5.186}$$

where $\lambda_c = \hbar/m_0 c$ is the reduced Compton wavelength of the particle. Solving equation (5.186) for v gives

$$v = \sqrt{1-(4\pi r_s/\lambda_c)^2}\ \ c, \tag{5.187}$$

which shows that for ordinary matter v is less than c provided that

$$(4\pi r_s/\lambda_c) < 1. \tag{5.188}$$

Therefore, black holes of mass M can produce particles of ordinary matter of mass m_0 only if equation (5.188) is satisfied. The flux of these particles will have a thermal spectrum as shown in equation (5.125).

In the case of tachyons $m = \frac{p_\infty/c}{\sqrt{(v/c)^2-1}}$ using Hill and Cox (2012) case I. Setting this equal to m in equation (5.184) we have

$$\frac{p_\infty/c}{\sqrt{(v/c)^2-1}} = \frac{\hbar}{4\pi\ c\ r_s}, \tag{5.189}$$

so that

$$4\pi r_s/\lambda_\infty = \sqrt{(v/c)^2-1}, \tag{5.190}$$

where $\lambda_\infty = \hbar/p_\infty$ and where, again, p_∞ is the value of the momentum in the limit as the relative velocity, v, approaches infinity. Solving equation (5.190) for v results in

$$v = \sqrt{1+(4\pi r_s/\lambda_\infty)^2}\ \ c. \tag{5.191}$$

As expected v is greater than c for tachyons. From equation (5.191) it is clear that the more massive the black hole (large $M \Rightarrow$ large r_s), the greater the tachyon velocity.

Finally, as explained preceding equation (5.183), this equation is the characteristic energy of particles created by a Schwarzschild black hole and its form is that of a thermal spectrum for the radiation flux as viewed by a distant observer.

5.7 Summary of Hawking Radiation for Higher-Derivative Klein-Gordon Equation

Hawking radiation for the ordinary Klein-Gordon equation is already well-known for both massless fields and massive fields and the results are pointed out in section 5.1 (massless fields) and section 5.3 (massive fields). The only difference between the results is that the ω for massive fields will include the rest mass as pointed out by Hawking (1975). For the higher-derivative Klein-Gordon equation the results in the case of the massless fields is simply twice the result of the ordinary Klein-Gordon equation for massless fields. In the case of massive fields, however, the higher-derivative Klein-Gordon equation factors into one part which is just the ordinary Klein-Gordon equation for massive fields and a second part which is the Klein-Gordon equation for tachyonic fields. It is this second part whose solutions offer new and interesting results. In particular, the solutions show that very near the event horizon the tachyonic velocity must be very close to c, the speed of light. For distances much greater than the event horizon, the velocity will be $v = \sqrt{1 + (4\pi r_s/\lambda_\infty)^2}\ \ c$, using Hill and Cox (2012). Since $r_s = 2GM/c^2$ and increases linearly as M increases, the tachyonic velocity will be large for large M.

CHAPTER 6

Nonlocality

6.1 Einstein/Podolsky/Rosen Paper

Before the formulation of quantum mechanics all physics was considered to be local, i.e. measuring a physical property of one of two different entities which had interacted with each other and then traveled away from each other would have no effect on the second entity. Furthermore, it was understood that one could measure any and all of the properties of a physical entity to as precise a degree as the instrumentation permitted. But outcomes of thought experiments and actual experiments suggested results that many physicists, including Einstein, found difficult to accept. This led to different interpretations of the results of experiments predicted by quantum mechanics and led to many debates among their respective adherents, most notably between Niels Bohr and Albert Einstein. The three principle interpretations were the Copenhagen interpretation (Bohr), the realist interpretation (Einstein) and, somewhat later, the agnostic interpretation.

In 1935 Einstein co-authored a paper with Boris Podolsky and Nathan Rosen (Einstein et al., 1935) in which they claimed by means of a thought experiment that quantum mechanics, although correct, was incomplete. Their argument was that for a theory to be complete "every element of the physical reality must have a counterpart in the physical theory" and that the "physical reality" of the system under consideration are those physical quantities which can be measured by experiments. They argued that according to quantum mechanics two operators, say A and B, which do not commute, i.e. $AB \neq BA$, then simultaneous knowledge of both of them is not possible which led to their conclusion "that either (1) the quantum description of reality given by the wave equation is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality." This result led Einstein to propose the "hidden variable" concept. According to this concept the quantum mechanical wave function needed further elaboration to predict the simultaneous existence of the properties of interest of the system under consideration. Einstein believed that all the properties of a system simultaneously existed but that the quantum mechanical theory was not yet comprehensive enough.

6.2 Bell's Inequalities

John Stewart Bell whose remunerative work was in particle physics and quantum field theory was also interested in the foundations of quantum theory and his best known and most influential paper was one addressing the Einstein/Podolsky/Rosen (EPR) paradox. In that paper (Bell, 1964), which was written in 1964, Bell shows that, although under very specific conditions of the Einstein proposal of hidden variables the predictions obtained from quantum mechanics can be reproduced, in general the hidden variables model will differ from the predictions of quantum mechanics. Bell does not use the precise thought experiment of EPR but rather a slight revision of it by David Bohm and Yakir Aharonov (Bohm and Aharonov, 1957). In their revision two particles are produced in an entangled state (spin singlet state) in which they move apart in opposite directions. The particles are detected by Stern-Gerlach magnets which are orientable in different directions. Each detector's measurement is either $+1$ or -1 depending on the spin of the particle measured. The orientation of the detectors is represented by a unit vector $\mathbf{a}$ for one apparatus and unit vector $\mathbf{b}$

for the other. The correlation between the instruments according to the prediction of quantum mechanics will be

$$P(\mathbf{a},\mathbf{b}) = -\mathbf{a}\cdot\mathbf{b}. \tag{6.1}$$

so if $\mathbf{a} = \mathbf{b}$, then $P(\mathbf{a},\mathbf{a}) = -1$. If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other $\mathbf{a}\cdot\mathbf{b} = 0$ and the measurements are uncorrelated. Bell gives an example showing that these special cases can be explained using hidden variables. However, he goes on to show that allowing for all possible orientations of the detectors hidden variables cannot explain the outcomes.

Bell proposed a hidden variable, λ, to account for the correlations between the two detectors. This hidden variable, λ, could be a simple variable, a set of variables or even a set of functions and could be continuous or discrete. Using λ the correlation between the two detectors would be in terms of an integral

$$P(\mathbf{a},\mathbf{b}) = \int d\lambda \rho(\lambda) A(\mathbf{a},\lambda), B(\mathbf{b},\lambda), \tag{6.2}$$

where $\rho(\lambda)$ is the probability density function, $A(\mathbf{a},\lambda)$ and $B(\mathbf{b},\lambda)$ are the measurement results of the respective detector, A or B, and

$$A(\mathbf{a},\lambda) = \pm 1, \quad B(\mathbf{b},\lambda) = \pm 1. \tag{6.3}$$

The two detectors, A and B, are physically separated from one another so the outcome on detector A does not depend on the setting $\mathbf{b}$ and likewise B does not depend on setting $\mathbf{a}$. Suppose the experimenter can set detector B to another setting different from $\mathbf{b}$, call it $\mathbf{c}$, a unit vector. Bell proves that $P(\mathbf{a},\mathbf{b})$ and $P(\mathbf{a},\mathbf{c})$ must satisfy the relation

$$|P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq 1 + P(\mathbf{b},\mathbf{c}). \tag{6.4}$$

As Griffiths (Griffiths, 2005) points out there are many settings for $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ for which a quantum mechanical result violates this inequality. One such case would be for **a** and **b** to be orthogonal and **c** to be in the plane of **a** and **b** but at a 45° angle to both of them. Then we would have

$$P(\mathbf{a},\mathbf{b}) = 0, \qquad P(\mathbf{a},\mathbf{c}) = P(\mathbf{b},\mathbf{c}) = -\frac{\sqrt{2}}{2}, \tag{6.5}$$

and from equation (6.4) this result would lead to an invalid inequality

$$\frac{\sqrt{2}}{2} \leq 1 - \frac{\sqrt{2}}{2}. \tag{6.6}$$

But clearly

$$\frac{\sqrt{2}}{2} \not\leq 1 - \frac{\sqrt{2}}{2}, \tag{6.7}$$

showing that no local hidden variable model can be consistent with the results of quantum mechanics for all possible settings of **a**, **b**, and **c**.

From this theoretical demonstration it appeared that Einstein was mistaken in his belief of local hidden variables. However, it would take many years and many experiments to substantiate Bell's inequality.

The problem with verifying Bell's inequality with experiment is that there were a number of loopholes to address which required ever more clever and sophisticated experiments. From 1970 onward there have been many notable experiments that have gradually closed these loopholes. The loopholes number six:

1) The detection loophole - Many photons are produced but few are detected leading to the possibility that those detected are unrepresentative even though they violate Bell's inequality. Other systems not using photons, e.g. trapped ions, have been more efficient. More recently, optical systems have been developed which have high efficiencies, such as superconducting photodetectors.

2) The locality loophole - There is one measurement for the property of interest

for each of the two particles being tested. To prevent the possibility of a signal being sent from the detector of the first particle tested to the second detector the two events (measurements) must have space-like separation. In 1982 Alain Aspect conducted an experiment which respected this condition and also allowed for the settings of each detector to be set during the flight of the particles. Gregor Weihs in 1998 improved on Aspect's experiment by devising a way for the settings to be determined randomly.

3) The coincidence loophole - The particle pairs produced by the source are numerous and some means is necessary to assure that the measurements made by the two detectors correspond to particles of the same pair.

4) The memory loophole - This loophole posits that, given that measurements are made with the same two detectors numerous times, somehow the hidden variables could use that information to increase the violation of Bell's inequality.

5) Superdeterminism - Superdeterminism asserts that free will does not exist and that the measurement settings are pre-determined by the system. This loophole would be impossible to falsify.

6) The many-worlds loophole - Bell's theorem assumes one single outcome to an experiment and so the many-worlds conception would not be consistent with that.

6.3 **The Nobel Prize in Physics 2022**

6.3.1 John Clauser

In 1972 John Clauser and Stuart Freedman used experimental equipment that had been built by a physicist Carl Kocher. However, they had to modify the equipment due to the inefficiencies of the polarizers but eventually they were able to establish a violation of the Bell-CHSH (Clauser-Horne-Shimony-Holt) inequality. Freedman died in 2012 and was therefore not named in the Nobel award, but the Freedman-Clauser experiment helped eliminate the coincidence loophole.

6.3.2 Alain Aspect

In 1981 and 1982 Aspect developed technics and specialized equipment to circumvent the locality loophole. He and his collaborators also used polarizers that changed settings during the flight of the particles. This experiment violated the Bell inequality and was many times more precise than the Freedman-Clauser experiment. But it was his third experiment performed in 1982 which was the most notable. Aspect used acousto-optical switches which could channel the photons into two different paths in a time of approximately 10 nanoseconds, which was shorter than the approximately 20 nanoseconds travel time of the photons. The results of these experiments violated Bell's inequality but were in agreement with quantum mechanics predictions.

6.3.3 Anton Zeilinger

Many years after Aspect's studies, in 1998 Zeilinger and his group refined Aspect's experiments and was able to use random numbers, and in one case signals from distant galaxies, to control the settings of the detectors. As a result of his work he demonstrated quantum teleportation and developed tools for use in quantum information and quantum computing.

6.4 Nonrelativistic Limit of the Fourth Order Higher-Derivative Klein-Gordon Equation

The following uses natural units for which $c = \hbar = 1$ and $m =$ rest mass.

The higher-derivative Klein-Gordon equation is

$$[(\partial^\mu \partial_\mu)^2 - m^4]\ \phi(\mathbf{r}, t) = 0. \tag{6.8}$$

Assume the form of $\phi(\mathbf{r}, t)$ is

$$\phi(\mathbf{r}, t) = \tilde{\phi}(\mathbf{r}, t)\ e^{-imt}, \tag{6.9}$$

and hence

$$[(\partial^\mu \partial_\mu)^2 - m^4]\ \tilde{\phi}(\mathbf{r}, t)\ e^{-imt} = 0. \tag{6.10}$$

Greiner (1990) shows that, if $E = E' + m^2$, where E is total energy and E' is the nonrelativistic kinetic energy, then $E' \ll m$ and

$$|i\partial_t \tilde{\phi}| \approx E'\tilde{\phi} \ll m\tilde{\phi}. \tag{6.11}$$

As an aid to what follows we note that from equation (6.11)

$$|\partial_t^2 \tilde{\phi}| \ll |2m\ \partial_t \tilde{\phi}|, \quad |\partial_t^3 \tilde{\phi}| \ll |2m\ \partial_t^2 \tilde{\phi}|, \quad |\partial_t^4 \tilde{\phi}| \ll |2m\ \partial_t^3 \tilde{\phi}|, \tag{6.12}$$

where we are simply repeatedly applying the differential operator to each side of the inequalities. But this implies that

$$|\partial_t^2 \tilde{\phi}| \ll |2m\ \partial_t \tilde{\phi}|, \quad |\partial_t^3 \tilde{\phi}| \ll |2m\ \partial_t \tilde{\phi}|, \quad |\partial_t^4 \tilde{\phi}| \ll |2m\ \partial_t \tilde{\phi}|. \tag{6.13}$$

Proceeding, we calculate the 2nd time derivative of ϕ

$$\begin{aligned}
\partial_t^2 \phi &= \partial_t \left[(\partial_t \tilde{\phi} - im\tilde{\phi})\ e^{-imt}\right] \\
&= (\partial_t^2 \tilde{\phi} - im\partial_t \tilde{\phi} - im\partial_t \tilde{\phi} - m^2 \tilde{\phi})\ e^{-imt} \\
&= (\partial_t^2 \tilde{\phi} - 2im\partial_t \tilde{\phi} - m^2 \tilde{\phi})\ e^{-imt} \\
&\approx (-2im\partial_t \tilde{\phi} - m^2 \tilde{\phi})\ e^{-imt}.
\end{aligned} \tag{6.14}$$

Now calculating the 3rd time derivative of ϕ we have

$$\partial_t^3 \phi = \partial_t \left[(\partial_t^2 \tilde{\phi} - 2im\partial_t \tilde{\phi} - m^2 \tilde{\phi})\ e^{-imt} \right]$$

$$= (\partial_t^3 \tilde{\phi} - im\partial_t^2 \tilde{\phi} - 2im\partial_t^2 \tilde{\phi} - 2m^2 \partial_t \tilde{\phi} - m^2 \partial_t \tilde{\phi} + im^3 \tilde{\phi})\ e^{-imt}$$

$$= (\partial_t^3 \tilde{\phi} - 3im\partial_t^2 \tilde{\phi} - 3m^2 \partial_t \tilde{\phi} + im^3 \tilde{\phi})\ e^{-imt}$$

$$\approx \left(-3m^2 \partial_t \tilde{\phi} + im^3 \tilde{\phi} \right)\ e^{imt}. \tag{6.15}$$

And finally we arrive at the 4th time derivative of ϕ

$$\partial_t^4 \phi = \partial_t \left[(\partial_t^3 \tilde{\phi} - 3im\partial_t^2 \tilde{\phi} - 3m^2 \partial_t \tilde{\phi} + im^3 \tilde{\phi})\ e^{-imt} \right]$$

$$= (\partial_t^4 \tilde{\phi} - im\partial_t^3 \tilde{\phi} - 3im\partial_t^3 \tilde{\phi} - 3m^2 \partial_t^2 \tilde{\phi} - 3m^2 \partial_t^2 \tilde{\phi} + 3im^3 \partial \tilde{\phi} + im^3 \partial \tilde{\phi} + m^4 \tilde{\phi})\ e^{-imt}$$

$$= (\partial_t^4 \tilde{\phi} - 4im\partial_t^3 \tilde{\phi} - 6m^2 \partial_t^2 \tilde{\phi} + 4im^3 \partial_t \tilde{\phi} + m^4 \tilde{\phi})\ e^{-imt}$$

$$\approx (4im^3 \partial_t \tilde{\phi} + m^4 \tilde{\phi})\ e^{-imt}. \tag{6.16}$$

Using equations (6.14), (6.15) and (6.16) we obtain the nonrelativistic limit of the H-D K-G equation

$$[(\partial^\mu \partial_\mu)^2 - m^4]\ \tilde{\phi}\ e^{-imt} = [(\partial_t^2 - \nabla^2)^2 - m^4]\ \tilde{\phi}\ e^{imt}$$

$$= (\partial_t^4 - 2\partial_t^2 \nabla^2 + \nabla^4 - m^4)\ \tilde{\phi}\ e^{imt}$$

$$\approx [(4im^3 \partial_t + m^4) - 2(-2im\partial_t - m^2)\nabla^2 + \nabla^4 - m^4]\ \tilde{\phi}\ e^{imt}$$

$$= [(4im^3 \partial_t - 2(-2im\partial_t - m^2)\nabla^2 + \nabla^4]\ \tilde{\phi}\ e^{imt} = 0. \tag{6.17}$$

Dividing by $4m^3 e^{imt}$ and collecting terms we have

$$[(i\partial_t + (i/m^2\ \partial_t + 1/2m)\nabla^2 + 1/4m^3\ \nabla^4]\ \tilde{\phi}$$

$$= [(i\partial_t + 1/2m\ \nabla^2 + i/m^2\ \partial_t\ \nabla^2 + 1/4m^3\ \nabla^4]\ \tilde{\phi} = 0, \tag{6.18}$$

or rewriting

$$\left[\left(i\partial_t + 1/2m\ \nabla^2 \right) + \left(i\ \partial_t + 1/4m\ \nabla^2 \right) \left(1/m^2\ \nabla^2 \right) \right]\ \tilde{\phi} = 0. \tag{6.19}$$

As an example, we use the mass of an electron, $m = 0.511$ MeV $= 0.511 \times 10^6$ eV. Then $\dfrac{1}{m^2} = 3.830 \times 10^{-12}$ and we have the relations

$$\left|\frac{i}{m^2}\nabla^2\partial_t\tilde{\phi}\right| \ll \left|i\partial_t\tilde{\phi}\right| \quad \text{and} \quad \left|\frac{1}{4m^3}(\nabla^2)^2\tilde{\phi}\right| \ll \left|\frac{1}{2m}\nabla^2\tilde{\phi}\right|.$$

Therefore, the second term in equation (6.19) is negligible compared to the first term, and the first term is just the Schrödinger equation.

6.5 A Possible Explanation For Nonlocality

As described in chapter 3 all odd power higher-derivative Klein-Gordon equations can be derived from, and is included in, even power equations since any odd power equation will appear by factoring an even power equation when $m = 2n$, where m is even and n is odd. The even power higher-derivative Klein-Gordon equation will be the product of three factors. One factor which is the Klein-Gordon equation and a second factor which we can refer to as the negative Klein-Gordon equation. Both these factors have real solutions for omega ($\omega^2 = k^2 \pm \omega_0^2$). The third factor has complex solutions for omega which are ignored on physical relevance grounds.

Also noted at the end of chapter 3 was that the wave function solutions to the higher-derivative Klein-Gordon (H-D K-G) equations could be interpreted as fields possessing a time-like dynamic and a space-like dynamic. All H-D K-G equations are Poincaré invariant (Musielak and Fry, 2009) and so their solutions must be Poincaré invariant and consistent with the four momentum relationship. The time-like dynamic ($\omega_{1\pm}$ solutions) is viewed as particles of ordinary matter with a wave function satisfying the higher-derivative Klein-Gordon equation and is responsible for the evolution of the Hamiltonian, in agreement with one of the axioms of quantum mechanics, i.e. that the time evolution of the wave function is governed by the Hamiltonian. These particles, again, satisfying the higher-derivative Klein-Gordon equation, also have solution that has a space-like dynamic ($\omega_{2\pm}$ solutions) which is governed by the distinctly quantum mechanical nature of matter, that is, the noncommutativity of observables. This space-like contact between all parts of the field would communicate the nonclassical state information not related to the Hamiltonian.

A field evolving with a space-like dynamic could explain the EPR thought experiment and Bell's inequality as well as communication between entangled particles separated by some distance. This communication between particles is not meant to imply faster than light particle particles or tachyons which are dismissed in this interpretation, but that the space-like aspect is responsible for the apparent communication between the particles. As a result it permits ordinary entangled particles to "communicate" with one another. However, the space-like part of the wave function would not be observable and would have no effect on the probability density which is determined solely by the time-like component of the wave function.

The fourth order K-G equation contains the original second order K-G equation and, as such, it describes spin-0 particles. Since the wave function field is based on the fourth order K-G equation it has a relativistic origin. As explained, the time-like and space-like dynamics are equally represented in the field. However, the space-like property does not persist in the nonrelativistic limit and the standard Schrödinger equation is recovered. The solutions to the H-D K-G equation provide an additional aspect of the wave function, a space-like dynamic, and this additional feature could be a possible explanation to the phenomenon of nonlocality.

CHAPTER 7

Summary

The early 20th century saw the development of quantum mechanics followed by the development of quantum field theory. In quantum field theory particle spin was of profound importance for formulating the fundamental equations describing particle dynamics; for spin-0 particles this was the Klein-Gordon equation. The higher derivative Klein-Gordon equations were constructed using irreducible representations of the Poincaré group. These equations were used to develop a higher derivative quantum field theory for spin-0 particles and tachyonic fields. It was noted that the increasingly higher orders of the Klein-Gordon equation with a mass term could be separated into two series of orders, an odd order series and an even order series. The odd order series can be factored into two factors, one of which is the original Klein-Gordon equation and another factor for which the energy (ω) has complex solutions and is therefore physically irrelevant. The even order series, when factored, yields three factors : a factor which is the original Klein-Gordon equation, a second factor which is the ”negative” Klein-Gordon equation (a minus sign rather than a plus sign preceding the mass term), and a third factor for which the energy (ω) has complex solutions and is therefore deemed physically irrelevant. It is the solutions to the ”negative” Klein-Gordon equation which seems to imply tachyonic, or space-like, fields.

Black holes are classified according to their mass, net electrical charge and angular momentum. A Schwarzschild black hole can have different masses but has no net electrical charge nor angular momentum. The black hole mass is inversely proportional to the energy available at the event horizon to produce the flux of emitted radiation. In other words, the smaller the black hole mass, the greater the energy available at the event horizon. As an example of what particles might be created and their likelihood of creation we note that for an intermediate mass black hole on the order of a solar mass, the event horizon temperature is $\sim 10^{-8}$ K. If we consider a solar mass black hole, then using the Stefan-Boltzmann law of blackbody radiation, the event horizon surface temperature in ordinary units is

$$T = \frac{\hbar\ c^3}{8\ \pi\ G\ k_B\ M_\odot} \cong 6 \times 10^{-8}\ \text{K},$$

where $\hbar$ is the reduced Planck constant, c is the speed of light, G is the gravitational constant, k_B is Boltzmann's constant and $M_\odot$ is the solar mass ($\sim 2 \times 10^{30}$ kg.). This represents an energy of

$$k_B T \cong 8 \times 10^{-31} \text{Joules},$$

whereas for the Higgs particle (the only known spin-0 particle) the mass energy is $\sim$ 125 GeV $\cong 2 \times 10^{-8}$ Joules, exceeding the energy at the event horizon of a solar mass black hole by roughly 23 orders of magnitude. Thus, the creation of a Higgs particle near the event horizon of a solar mass black hole is essentially zero. However, if the black hole is very small, say on the order of 10^8 kg, then the event horizon temperature is orders of magnitude higher so that the energy at the event horizon is comparable to the Higgs mass energy and there is the likelihood of the creation of Higgs particles.

Application of the "negative" equation to a Schwarzschild black hole adds tachyonic particles or fields to the usual Hawking radiation as energy radiated from the black hole by quantum mechanical effects. A distant observer cannot see the creation of the particles but does see a continuous flux of particles and is able to measure their speeds. The distant observer measures a characteristic energy of any of the particles generated by a black hole as

$$\mathcal{E} = \frac{\hbar\ c^3}{8\pi\ G\ M}$$

Setting this equation equal to the equation for energy of particle in the superluminal regime

$$\mathcal{E} = \frac{p_\infty c}{\sqrt{(v/c)^2 - 1}},$$

where p_∞ is the (finite) value of the momentum as the relative velocity, v, approaches infinity, yields

$$\mathcal{E} = \frac{p_\infty c}{\sqrt{(v/c)^2 - 1}} = \frac{\hbar\ c^3}{8\pi\ G\ M}$$

and

$$v = \left[1 + \left(\frac{8\pi G M p_\infty}{\hbar c^2}\right)^2\right]\ c.$$

It is seen that the tachyon velocity is greater than c and that the more massive the black hole, the greater the tachyon velocity, as opposed to the relation between black hole mass and particle velocity of ordinary matter. All this for a distant observer.

The wave function solutions to the higher-derivative Klein-Gordon equations allow the identification of those solutions as fields possessing a time-like dynamic and a space-like dynamic which are represented equally in the field. The time-like evolution of the wave function is governed by the Hamiltonian and operates in the subluminal domain so there is no violation of causality. The space-like dynamic is a strictly quantum mechanical nature of matter, i.e. the noncommutitivity of observables. A possible explanation for the phenomenon of nonlocality is the space-like property of the wave function, although it would not be observable and would have no effect on the probability density.

Finally, the higher derivative Klein-Gordon equations are relativistic equations, but the nonrelativistic limit recovers the Schrödinger equation.

CHAPTER 8

Addendum - Convergence of Integrals

We need to show the convergence of integrals for equations (5.87) and (5.88). Consider the integrals in the lower-half complex plane for $\alpha_{2\omega\omega'}$ (equation (5.87)) on the infinite quarter circle in the 4th quadrant and for $\beta_{2\omega\omega'}$ (equation (5.88)) in the 3rd quadrant. The integral for $\alpha_{2\omega\omega'}$ is

$$\alpha_{2\omega\omega'} = -C \int_{\infty}^{0} ds \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} r^2 \; e^{-i\omega' s} \; e^{i\omega' v_0} \; e^{i\omega 4M \ln(\frac{s}{K})}.$$

This integral is equal to $-C \int_{\infty}^{0} ds \; (\frac{\omega'}{\omega})^{\frac{1}{2}} \; r^2 \; e^{-i\omega' s} \; e^{i\omega' v_0} \; e^{i\omega 4M(\ln s - \ln K)}$.
But $(\frac{\omega'}{\omega})^{\frac{1}{2}}$, $e^{i\omega' v_0}$ and $e^{-i\omega 4M \ln K}$ are all constants so the integral can be written

$$\alpha_{2\omega\omega'} = C' \int_{0}^{\infty} ds \; r^2 \; e^{-i\omega' s} \; e^{i\omega 4M \ln s}, \text{ where } C' \text{ is a constant.}$$

The factor $e^{i\omega 4M \ln s}$ is equal to $e^{i\omega 4M \ln z}$ where $z = Re^{i\theta}$ (since we are in the complex plane) and the absolute value of this factor is

$$|e^{i\omega 4M \ln z}| = |e^{i\omega 4M \ln(Re^{i\theta})}| = |e^{i\omega 4M(\ln R + i\theta)}| = |e^{i\omega 4M \ln R}| e^{-\omega 4M\theta} \leq e^{2\pi\omega M},$$

since $-\frac{\pi}{2} \leq \theta \leq 0$ (4th quadrant).
Now consider the factor $e^{-i\omega' s} = e^{-i\omega' z} = e^{-i\omega' Re^{i\theta}} = e^{-i\omega R\cos\theta + \omega R \sin\theta} = e^{-i\omega R\cos\theta} e^{\omega R \sin\theta}$.
We have $|e^{-i\omega R\cos\theta}| \leq 1$ but $e^{\omega R\sin\theta} \to 0$ as $R \to \infty$ (since $\sin\theta < 0$ in the 4th quadrant). (This R is not to be confused with $R(r)$ from equations (5.74) through (5.79)). So $r^2 e^{\omega R \sin\theta} \to 0$ as $R \to \infty$ for any finite $r > 2M$. Therefore, the integral of $\alpha_{2\omega\omega'}$ converges on the boundary at infinity in the 4th quadrant. A similar analysis of the integral for $\beta_{2\omega\omega'}$ shows that it converges on the boundary at infinity in the 3rd quadrant.

Bibliography

Bagla, K.S., Jassal, H.K., Padmanabhan, T., 2003, Physical Review D, 67, 063504

Bell, J.S., 1964, Physics 1, 195-200

Bilaniuk, O.-M., Sudarshan, E. C. G.,1969, Physics Today, May 1969, 43-51

Birrell, N.D., Davies, P.C.W., 1982, Quantum Fields in Curved Space (Cambridge University Press)

Bohm, D., Aharonov, Y., 1957, Phys. Rev. 108, 1070

Carroll, Sean M., 2004, An Introduction to General Relativity Spacetime and Geometry, 1st edn. (Addison Wesley)

de Broglie, L., 1925, Ann. Phys. (Paris) 3, 22.

de Urries, F.J., Julve, J., 1998, arXiv:hep-th/9802115v2 24 March 1998

Dirac, P.A.M., 1928, Proc. Roy. Soc. (London) A117 (1928) 610; A118 (1928) 351.

Einstein, A., 1905, Ann. der Physik 18, 639

Einstein, A., Podolsky, B., Rosen, N., 1935, Physical Review, 47, 777-780

Feinberg, G., 1967, Physical Review, 159, 1089-1105

Fry, J.L. and Musielak, Z.E., 2010, Ann. Phys. 325, 2668

Fry, J.L., Musielak, Z.E., Trei-wen Chang, 2011, Ann. Phys., 326, 1972

Fulling, S.A., 1989, Aspects of Quantum Field Theory in Curved Space-Time (Cambridge University Press)

Gordon, W., 1926, Zeits. für Phys. 40, 117; 40, 121.

Greiner, W., 1990, Relativistic Quantum Mechanics (Springer-Verlag)

Greiner, W., Reinhardt, J., 1996, Field Quantization (Springer-Verlag)

Griffith, D.J., 2005, Introduction to Quantum Mechanics, 420-428, (Pearson Prentice Hall)

Ha, Y. K., 2003, General Relativity and Gravitation, 35, 2045-2050

Hawking, S., 1971, Monthly Notices of the Royal Astronomical Society, 152:75

Hawking, S.W., 1975, Commun. Math. Phys., 43, 199

He, X-G., Ma, B-Q., 2011, Mod. Phys. Lett. A, 26, 2299 [arXiv:1003.2510 [hep-th]]

Hill, J. M., Cox, B. J., 2012, Proc. R. Soc. A, 468, 4174

Hobson, M.P., Efstathiou, G., Lasenby, A.N., 2006, General Relativity (Cambridge University Press)

Kim, Y.S., Noz, M.E., 1986, Theory and Applications of the Poincaé Group (D. Reidel Publishing Company)

Klein, O., Z., 1926 Phys. 37, 895.

Merzbacher, E., 1997, Quantum Mechanics, Wiley, New York

Mignani, R. and Recami, E., 1976, Lettere al Nuovo Cimento (1971-1985) 16(15), 449-452

Minkowski, A., 1908, Space and Time, delivered at the 80th Assembly of German Scientists and Physicians at Cologne, 21 Sept. 1908

Musielak, Z.E. and Fry, J.L., 2009, Ann. Phys. 324, 296

Musielak, Z. E., Fry, J. L., 2009, Int. J. Theor. Phys., 48, 1194

Musielak, Z.E., Fry, J.L., Kanan, G.W., 2015, Adv. St. in Theor. Phys., 9, 213

Parker, L. E., Toms, D. J., 2009, Quantum Field Theory in Curved Spacetime (Cambridge University Press)

Pauli, W., 1927, Z. Physik 43, 601.

Planck, M., 1901, Ann. Phys. 309, 553

Podolski, B., 1942, Phys. Rev, 62, 68

Proca, A., 1936, Le Journal de Physique et le Radium, 7, 347

Ryder, Lewis H., 1996, Quantum Field Theory (Cambridge University Press)

Sahoo, S., Kumar, M., 2012, ARPN J. of Sci. and Tech., 2, 182

Schrödinger, E., 1926, Phys. Rev. 28, 1049

Schwartz, C., 1982, Physical Review D, 25, 356-364

Schwartz, C., 2011, J. Math. Phys., 52, 052501

Schwartz, C., 2016, Internat. J. Mod. Phys. A, 31, 1650041

Srivastava, S. K., 1983, J. Math. Phys., 24, 1317

Susskind, L., 1994, SU-ITP-94-33, September 1994

Weldon, H. A., 2003, Ann.Phys., 305, 137

Whittaker, E.T., 1947, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Cambridge University Press)

Wigner, E.P., 1939, Ann. Math., 40, 149

Zel'dovitch, Y.B., Novikov, I.D., 1966, The Hypothesis of Cover Retarded During Expansion and the Hot Cosmological Model, Soviet Astronomy, 10(4), 602-603